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Reprinted from AIChE JOURNAL, September 1970

COMMUNICATIONS TO THE EDITOR

The Effect of Property Variations on the Convective Instability of Gases

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The onset of convection in a fluid layer heated from below is governed by equations which are usually simplified by using the Boussinesq approximation: the fluid is incompressible; the viscosity, thermal conductivity, and specific heat are constant; and the variation of density with temperature is neglected except in the gravitational term that drives the motion. This assumption is certainly justified for liquids which usually have property variations of less than 1%. In gases, however, property variations often are as large as 5%, and in some liquids, such as silicon oil, the viscosity variation can be even higher.

Palm (*7*) and Segel and Stuart (*9*) showed that when viscosity variations were taken into account, the Rayleigh number, according to linear theory, was modified by the following formula:

$$\frac{N_{Ra}}{(N_{Ra})_0} = 1 - 0.048 \left(\frac{\Delta\nu}{\nu_0}\right)^2 \qquad (1)$$

Thus the correction is second order in the dimensionless viscosity variation. Busse (*1*) has shown that the correction is second order when variations in any of the physical properties are allowed. Equation (1) was derived for the case of a fluid layer bounded by two free surfaces on which the temperature is fixed. Only in the case of the viscosity have the numerical values of the second order coefficients been reported.*

Instability analyses are being used more frequently in the chemical engineering literature (*5, 6, 8, 11* to *13*) in situations which require assumptions, similar to the Boussinesq approximation, for ease in analysis. The effect of assuming certain terms constant is not always clear, but it is reported below for the convective instability of gases contained between two rigid boundaries. A conventional linearized analysis is employed (*2*) and calculations are made by using the Galerkin method, which also provides guidelines to decide when property variations are important.

The fluid properties are allowed to vary with temperature according to a linear relation

$$a_i = a_{i0} + b_i(T - T_0) \qquad (2)$$

where $i = 2$, 3, and 4 correspond to viscosity, thermal conductivity, and specific heat, respectively. The temperature dependence of the thermal conductivity causes the temperature profile at the quiescent state to deviate from a linear dependence in vertical distance. With the temperature specified at $z = \pm ½$ (where z is the dimensionless vertical distance), the quiescent state temperature is given to second order in the small parameter b_3 by

TABLE 1. COEFFICIENT α_{ij}

i	j	α_{ij}	i	j	α_{ij}	i	j	α_{ij}
1	1	−0.0019	1	2	0.023	2	5	−0.066
2	2	−0.052	1	3	0.059	3	4	0.059
3	3	0.027	1	4	−0.0030	3	5	−0.17
4	4	−0.0019	2	3	−0.11	4	5	0.06
5	5	−0.14	2	4	0.023			

$$T^*(z) = \frac{T_0}{\beta d} - \left[1 - \frac{1}{8}\left(\frac{b_3\beta d}{a_{30}}\right)^2\right] z - \left(\frac{b_3\beta d}{a_{30}}\right)\frac{z^2}{2} - \left(\frac{b_3\beta d}{a_{30}}\right)^2 \frac{z^3}{2} \qquad (3)$$

This temperature distribution induces spatial variations in viscosity, thermal conductivity, specific heat, and density which affect the convective motion.

The theory applies to gases, in which case the equation of state is given by the perfect gas law

$$p/\rho = \hat{R}T \qquad (4)$$

and for liquids, in which case the density varies according to Equation (2) with $i = 1$. After linearizing the momentum and energy equations, assuming separation of variables, agreeing to consider only stationary instability, retaining only second-order terms in b_i, and recognizing that the fluid remains essentially incompressible and pressure variations are always small compared with temperature variations provided $gb_1d/a_1\hat{R}\sqrt{N_{Ra}} << 1$ and $g/\beta C_p << 1$,† we can reduce the dimensionless momentum and energy equations to

$$\nabla^2(\mu^*\nabla^2W) + 2\alpha^2(D^2\mu^*)W = N_{Ra}^{½}\,\frac{\rho^*}{T^*}\,\alpha\,\theta \qquad (5)$$

$$\nabla\cdot(k^*\nabla\,\theta) + B_3\nabla\cdot(\theta\,\nabla T^*) = N_{Ra}^{½}\,\alpha\,\rho^*C_p^*\,DT^*W \qquad (6)$$

which must be solved subject to the boundary conditions

$$W = DW = \theta = 0 \quad \text{at} \quad z = \pm ½ \qquad (7)$$

The effect of property variation is represented by the functions ρ^*, c_p^*, μ^*, k^*, and T^* which represent the spatial distributions of these properties due to the quies-

* Some numerical results are available for first-order corrections to fluid properties which give rise to second-order corrections to the Rayleigh number (*14* to *16*). These analyses did not consider second-order corrections, which also give rise to second-order corrections to the Rayleigh number.

† The small effect of compressibility is discussed in references *16* to *18*.

cent state temperature distribution. When there are no property variations, these terms are constant, except for T^* which is linear in z, and the equations reduce to those appropriate to the Boussinesq approximation. For the case of property variations, then, the problem reduces to an eigenvalue problem with variable coefficients.

The equations are solved by using the Galerkin method with the following trial functions:

$$W_k = z^{k-1}\left(\frac{1}{4} - z^2\right)^2$$
$$\theta_k = z^{k-1}\left(\frac{1}{4} - z^2\right) \tag{8}$$

[See Finlayson (3) for a description of the Galerkin method.] Only stationary instability was considered, since the results of Davis (4) suggest that oscillatory instability cannot occur. All calculations were done by using four terms in the expansion for W and θ. For the case of no property variations, this gives $(N_{Ra})_0 = 1{,}709$ compared with the exact value of 1,707.765 (10). Six terms gave $(N_{Ra})_0 = 1{,}707.77$, but the improved accuracy was deemed unimportant in this study so that only four terms were used. When the properties are constant, of course, W and θ are even functions, and only even terms need to be retained in Equation (8). The approximations 1,709 and 1,707.77 then refer to two- and three-term expansions, respectively.

For small B_i, the critical Rayleigh number follows the relation

$$\frac{N_{Ra}}{(N_{Ra})_0} = 1 + \sum_{i=1}^{5}\sum_{j=1}^{i} \alpha_{ij}\, B_i\, B_j \tag{9}$$

where the coefficients α_{ij} are listed in Table 1 and the dimensionless coefficient of property variation is defined as

$$B_i = \frac{b_i \beta d}{a_{i0}} \quad i = 1, 2, 3, 4 \quad B_5 = \frac{\beta d}{T_0} \tag{10}$$

The effect of property variations is very small, since, for example, a viscosity variation of 10% induces a change in the critical Rayleigh number of only 0.05%. Notice that α_{22} is very close to the value of -0.048 determined by Palm and Segel for free boundaries, suggesting that these coefficients are not very sensitive to the type of boundary conditions.

In order to generalize the results, it is instructive to examine why first-order changes in property variations induce only second-order change in the Rayleigh number. All terms involving property variations in Equations (5) and (6) are of the form

$$a_i^* = 1 + 0(B_j)z + 0(B_j B_k)z^2 \tag{11}$$

In the Galerkin method, the first approximating function is an even function of z, corresponding to the approximation for the solution with constant physical properties. Since linear terms in the property variations occur only in conjunction with odd powers of z, it is clear that the first approximating function and these linear terms will not contribute to a change of Rayleigh number in the first approximation. In the second approximation, however, the approximating functions contain odd powers of z. These can interact with the terms linear in B_j, but a detailed examination of the 4×4 determinant arising in the Galerkin method indicates that the determinant is a function only of combinations of $B_i B_j$. The effect on the Rayleigh number must be second order. Consequently, in similar convective instability problems, whenever the solution to the eigenvalue problem with constant coefficients is an even function and the linear variations of the variable coefficients occur as odd functions, then including these variable coefficients induces a second-order effect on the eigenvalue. A similar conclusion can be reached by using the perturbation method.

ACKNOWLEDGMENT

This work was supported in part by the Office of Engineering Research, University of Washington.

NOTATION

a_i = physical property
b_i = temperature dependence of physical property, defined in Equation (2)
B_i = dimensionless physical property variation, Equation (10)
C_p = heat capacity
d = thickness of fluid layer
g = acceleration of gravity
k = thermal conductivity
N_{Ra} = $g(-b_1)\beta d^4/\rho_0\nu_0\kappa_0$, Rayleigh number
p = pressure
$\hat{R}$ = gas constant, per pound mass
T = temperature
W = dimensionless vertical velocity
z = vertical distance

Greek Letters

α = wave number, dimensionless
β = temperature gradient, $\Delta T/d$
θ = dimensionless temperature
κ = $k/\rho C_p$, thermal diffusivity
μ = viscosity
ν = μ/ρ, kinematic viscosity
ρ = density

Subscripts

0 = property evaluated at centerline temperature ($z = 0$)
i = $i = 1, 2, 3, 4$ means ρ, μ, k, C_p, respectively
k = denotes term in expansion functions for W and θ, Equation (8)

Superscript

$*$ = dimensionless function of position

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