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ORTHOGONAL COLLOCATION ON FINITE ELEMENTS — PROGRESS AND POTENTIAL *

Bruce A. FINLAYSON
Department of Chemical Engineering, University of Washington, Seattle, WA, U.S.A.

Abstract

The method of orthogonal collocation on finite elements is described for solution of ordinary and partial differential equations. Benefits and limitations of the method are outlined by comparison with Galerkin finite element methods. Practical difficulties are given which arise in the application to engineering problems. Areas for future research are suggested.

Introduction

In the collocation method the unknown solution is expanded in a series of functions, often polynomials, which include unknown parameters. The expansion is substituted into the differential equation to form the residual. The residual is set to zero at a set of points, called collocation points, and these equations provide the criteria needed to determine the parameters in the expansion. The collocation method has been used in engineering analysis since the 1930's [1], but usually with the trial function defined globally, i.e., over the entire domain. In the 1960's the Galerkin finite element method came into widespread use. This method differs in two respects: the Galerkin criterion is used rather than setting the residual to zero, and the trial functions are usually piecewise polynomials, defined over small regions of space, called elements, and zero elsewhere. In the 1970's this idea of finite elements was applied to collocation methods.

The promise of the collocation finite element method is that the equations are usually easier to set up and solve than in Galerkin methods. The convergence properties of the two methods are comparable, and the collocation method may lead to a more sparse matrix than that given by the Galerkin method. Difficulties do arise: the first possible trial function is a C^1 cubic, and such expansion functions are not appropriate for all problems, and may be inefficient for engineering accuracy. Meshes are not as easily distorted to handle irregular geometries as for isoparametric elements in the Galerkin method. Mesh refinement is not as easy because triangular elements have not been utilized with collocation methods. Some of these difficulties are just now being alleviated in ongoing research projects.

One-Dimensional Problems

Take the one-dimensional domain $0 \leqslant x \leqslant 1$ and discretize it by the knots,

$$x_1 = 0,\ x_2,\ x_3,\ \ldots,\ x_{NE},\ x_{NE+1} = 1$$

The k-th element is contained between x_k and x_{k+1}. On that element we define a local coordinate system using

$$\xi = (x - x_k)/h_k,\quad h_k = x_{k+1} - x_k$$

$$h = \max_k h_k$$

A collocation point or solution at that point in the local coordinate system is denoted by a capital letter, $_I$, while the same point, but in the global numbering system is denoted by a lower case letter, $_i$. Thus the function c_I in the k-th element is also c_i. The conversion from I to i and vice versa is understood and depends on the element number.

Within an element we interpolate a function u(x) by Hermite polynomials.

$$u = \sum_{I=1}^{NP} C_i H_I(\xi)$$

$$H_1 = (1 - \xi)^2\,(1 + 2\xi)$$

$$H_2 = (1 - \xi)^2\,\xi h_k$$

$$H_i = (1 - \xi)^2\,\xi^{i-1},\quad i = 3,\ \ldots,\ NP-2$$

$$H_{NP-1} = \xi^2(3 - 2\xi)$$

$$H_{NP} = \xi^2(\xi - 1)h_k$$

These functions are arranged so that they have zero slope and function at each end of the element, ξ=0,1, except for one such value. H_1 is 1 at $\xi = 0$, H_2 has a unit x- derivative at $\xi = 0$, H_3 is 1 at $\xi = 1$ and H_4 has a unit x- derivative at $\xi = 1$. Thus the coefficients C_i in the expansion represent the value of the function u or its derivative at the points $\xi = 0$ or 1.

Lagrangian interpolation can also be used.

$$u = \sum_{I=1}^{NP} C_i L_I(\xi)$$

$$L_I = \prod_{J=1,J\neq I}^{NP} (\xi-\xi_J) \Big/ \prod_{J=1,J\neq I}^{NP} (\xi_I - \xi_J)$$

Using NP = 4 gives a cubic polynomial with both interpolants. Use of the Hermite polynomials results in an interpolation that is in C^1 for $x\in[0,1]$ whereas the Lagrangian interpolation is not automatically in C^1. The first Hermite polynomial is a cubic function of position, while the first Lagrangian polynomial (for second order equations) is a quadratic function of position. This is in contrast to Galerkin methods which can employ linear functions on elements.

Still another alternative is to use an expansion function with B-splines and collocate at the knots, including the end points. If cubic B-splines are used then the expansion is in C^2 since first and second derivatives are continuous at the knots. This approach has been advocated by Sincovec [2] and de Boor [3].

Steady-State Problems

A typical steady state problem is

$$\frac{d}{dx}\left(\eta(u,\frac{du}{dx})\,\frac{du}{dx}\right) = f\left(u,\frac{du}{dx}\right) \qquad (1)$$

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The collocation method applied to this equation gives the following equations:

$$\sum_K A_{IK}\,\eta_K \sum_J A_{KJ}u_J = f(u_I, \sum_J A_{IJ}u_J) \qquad (2)$$

where $\left.\frac{du}{dx}\right|_I \equiv \sum_J A_{IJ}u_J$

The result is a set of possibly nonlinear algebraic equations which must be solved to find the approximate solution. The collocation criterion is applied at the Gauss quadrature points interior to each element; for cubic functions we have two such points for each element. We also have two boundary conditions, and this provides enough conditions to define the Hermite expansion. The Lagrangian interpolation has more parameters, however, ((NP-1)NE+1), and another condition must be applied on each element.

The added collocation condition can be obtained in two ways. Clearly we cannot collocate at the knot since the first and second derivatives are not necessarily continuous there, so that the second derivative may not be defined. The first approach was by Carey and Finlayson [4]: they appended a condition that the interpolation have continuous first derivatives at the knots. The solution is then in C^1; indeed it is the same solution as found with Hermite polynomials since it satisfies the same conditions. With Lagrangian cubics, however, approximately 3NE terms are needed, whereas for Hermite cubics only 2NE terms are needed. The Hermite solution thus involves less computational effort to solve.

The coefficient η in Eq. (1) can represent a thermal conductivity in a heat transfer problem, a diffusivity in a mass transfer problem, or a viscosity in a fluid flow problem. If such problems involve more than one material, these coefficients may be discontinuous at certain points in the domain $0 < x < 1$. In such cases the fluxes are usually continuous, i.e., $\eta du/dx$ is continuous across the region of changing material, although du/dx is not continuous there. For these problems the Lagrangian formulation is easily changed to allow for continuous fluxes, whereas the Hermite formulation is not so accommodating.

A recent addition to the scope of techniques available is to use Lagrangian interpolants but replace the slope condition at the knots by a Galerkin-type criterion [5, 6]. We define the "hat" functions as piecewise linear functions, with the l_k function centered at the k-th knot. Eq. (1) is multiplied by this "hat" function and the result integrated by parts as in a Galerkin method.

$$-\int_0^1 \eta \frac{d\ell_k}{dx}\frac{du}{dx}\,dx = \int_0^1 \ell_k f(u, du/dx)dx \qquad (3)$$

This equation, one for each interior knot, replaces the slope condition at the knots. The approximate solution is then in C^0, and the method is described as a C^0 Collocation-Galerkin method. The author prefers to call it a C^0 collocation-residual method, since Galerkin used the same functions for trial functions and weighting functions, which is not done in Eq. (3). Such a method may not have any apparent advantages for this problem, but it does for time-dependent problems and for two-dimensional problems.

Time-Dependent Problems

We next append a time derivative to Eq. (1) to obtain a parabolic partial differential equation.

$$g(u)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\eta\left(u, \frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial x}\right] - f\left(u, \frac{\partial u}{\partial x}\right) \qquad (4)$$

For steady-state problems the solution methods are all sufficiently fast that the efficiency of the method is usually not of great concern. For transient problems, however, many time steps must be taken, so that efficiency is important. Clearly the Hermite interpolant, which involves fewer unknowns per element, is more efficient, in that the resulting matrix problem can be solved faster.

There is one important reservation, however. Hermite interpolation applied to Eq. (4) gives

$$g(u_I)\sum_J C_{IJ}\frac{du_J}{dt} = \ldots$$

where the matrix C_{IJ} is a 2 x 4 matrix. Lagrangian interpolation, however, gives equations of the form

$$g(u_j)\frac{du_j}{dt} = \ldots$$

and the time operator is clearly diagonal. The C^0 Collocation-Weighted Residual method gives a time operator which is diagonal for the interior collocation points but is not diagonal for the knots. This means that neither Hermite collocation nor C^0 Collocation-Weighted Residual methods can use explicit integration schemes, such as Runge-Kutta, without additional processing. Lagrangian collocation methods have no such limitation. When the time derivative is linear, however, an LU decomposition of the time operator for the Hermite interpolation need only be done once per problem, so that most of the economy of the method is retained. Such limitations are not important if one uses an implicit integration in time, since then a matrix problem must be solved in any case.

None of the methods can be adapted to use integration packages which handle the time integration. Packages such as GEARB [7] are made to integrate problems in the form

$$\frac{dy_j}{dt} = f_j(y_1, \text{---}, y_n)$$

and must be revised for problems in the form

$$\sum C_{ji}(y_1, \text{---}, y_n)\frac{dy_i}{dt} = f_j(y_1, \text{---}, y_n).$$

Such revisions are necessary for Hermite interpolation or C^0 Collocation-Weighted Residual methods, and other revisions are necessary for Lagrangian interpolation. The use of Lagrangian interpolation gives a set of ordinary differential equations in time with some linear algebraic constraints. These constraints must be handled adroitly to avoid changing the band structure of the matrix. None of these changes are insurmountable; but they must be made to use a collocation method with a standard integration package. Similar problems involving the time derivative also occur for all Galerkin finite element methods. Madsen and Sincovec [8] have prepared a package which uses B-spline interpolation for the spatial dependence and a version of GEARB for the time dependence. The user then must only provide a subroutine defining the differential equation and boundary conditions. When such a package is not available, it is usually much simpler to apply finite difference methods to the partial differential equation, and then use an integration package to integrate in time. The finite difference method may not give the most efficient method when looking at

computer time alone, but when including programming time it may be the method of choice for "one-shot" one-dimensional applications.

Convergence

The convergence of collocation finite element methods is one area in which the mathematical theory outstrips the practice. Douglas [9] treated the linear problem (4) with $g = \eta = 1$ and $f = 0$. He showed that if 4th degree polynomials are used in space and second degree in time, and collocation is applied in both space and time, that the pointwise error at the knots in both space and time decreases with spatial step size h and temporal step size Δt as

$$\text{error} \alpha O(h^6) + O(\Delta t^4)$$

This is a special case of a more general result by Douglas and Dupont [10] for nonlinear equations of the form

$$g(u) \frac{\partial u}{\partial t} = a(u) \frac{\partial^2 u}{\partial x^2} + b\left(u, \frac{\partial u}{\partial x}\right)$$

They show that the error at the knots in space and time decreases with step size as

$$\text{error} \alpha O(h^{2r-2}) + O(\Delta t^{2s})$$

where r-degree polynomials are used in space and s-degree in time. For cubic polynomials in space this gives h^4.

deBoor and Swartz [11] treated collocation methods for steady state problems of the type

$$\frac{d^m u}{dx^m} = f\left(u, \frac{du}{dx}, \text{---}, \frac{d^{m-1} u}{dx^{m-1}}\right)$$

The polynomials are taken as degree $m + k - 1$ and the error at any point in the domain obeys

$$\text{global error} \alpha O(h^{m+k})$$

whereas the error at the knot obeys

$$\text{error at knots} \alpha O(h^{2k})$$

Such results are called superconvergence results since the rate of convergence at the knots is faster than globally. Consequently, the solution is expected to be more accurate at the knots. This result holds for arbitrary order of equation, and the analog of the result for initial value problems had been proved earlier by Hulme [12]. For second order equations and cubic polynomials the error goes as h^4 both globally and at the nodes.

For the C^0 Collocation-Weighted Residual method Dunn and Wheeler [13] have shown for a linear two-point boundary value problem that the pointwise error decreases as

$$\text{global error} \alpha O(h^s)$$

where s is bounded between 3 and r + 1, for polynomials of degree r. The exact solution has finite derivatives of at least degree s. For a problem in which the exact solution has bounded, fourth derivatives and cubic polynomials are used the error goes as h^4.

While the convergence theorems depend on various conditions on the coefficients in the equations, boundary conditions, etc., and may not apply to a problem of specific interest, they are general enough to provide guidelines for all cases and useful results for for some cases.

The error reduces very rapidly as the number of elements is increased, or as h decreases. One important limit on the convergence theorems is that they require the solution to be highly continuous. If the solution is not continuous, or does not have continuous derivatives, then the rate of convergence is not as rapid and is limited by the continuity of the solution rather than the degree of polynomial. The work to apply the methods goes up rapidly with high order methods. For example a finite difference method or Galerkin method with linear trial functions requires approximately 5n multiplications to solve a linear tritiagonal system, where n is the number of grid points. A Hermite polynomial cubic polynomial requires 12n multiplications to solve the linear matrix problem, for n elements. Consequently, the extra work must result in better accuracy if high order methods are to be efficient. In such cases, with exact solutions having discontinuous derivatives, the low order methods may be preferable, and in that case the collocations methods are inappropriate: cubic (or quadratic Lagrangian) polynomials are the lowest order polynomials possible. The collocation method may be as good or better than other methods which use cubic polynomials, such as the Galerkin method with cubic polynomials, but may not be competitive with methods which can use lower order polynomials, such as the Galerkin method (linear polynomials) or the finite difference method.

Applications

Early applications of OCFE in engineering were to heat and mass transfer problems. Carey and Finlayson [4] used Lagrangian interpolation for a reaction-diffusion problem (1). For large reaction rates, the concentration solution had a steep profile, or a large gradient, and small elements were necessary in that region to solve the problem. Mesh refinement was done using a large residual as a criterion for needing additional elements. Chawla et al. [14] used Hermite cubics for a nonlinear transient heat conduction problem (4). Finlayson [15] solved for water transport in dry soils using OCFE and found the method worked well only when small elements were used near the steepfront. More recent applications have been primarily in chemical engineering, usually for problems involving diffusion and reaction. Young and Finlayson [16] treated a monolith reactor, with reaction on the wall. For carbon monoxide oxidation the concentration and temperature profiles are steep and finite elements are necessary. Lee and Aris [17] treated a similar problem, but added a non-linear integral term to the equation to account for radiation. Birnbaum and Lapidus [18] have compared OCFE favorably to semi-implicit Runge-Kutta methods for solving boundary value problems (1) with diffusion and reaction. OCFE was several orders of magnitude faster than shooting methods for the problems treated. Hopkins and Wait [19] compared Galerkin and collocation finite element methods with finite difference methods for a variety of linear and nonlinear parabolic equations (4). They concluded that finite difference methods were faster than low-order Galerkin methods, which in turn were faster than high-order collocation methods, which were faster than high-order Galerkin methods. Unfortunately, the authors based comparisons on a similar number of unknowns, and the computation times varied widely. Indeed, detailed examination of the results indicates that the high-order results are not changed as more elements are added, and this means the error is primarily time truncation error rather than spatial error. The results do indicate, however, that low-order

methods may be better for some problems, and this leaves out collocation methods which are automatically high-order. Jensen and Finlayson [20] have applied OCFE to the convective diffusion equation.

$$\frac{\partial c}{\partial t} + Pe \frac{\partial c}{\partial x} = \frac{\partial^2 c}{\partial x^2}$$

When the Peclet number is large the equation is more nearly hyperbolic in character than parabolic. The solutions can then exhibit steep fronts which persist. If large elements are used the solutions oscillate unnaturally. OCFE has been no more successful in solving this problem than other methods. Small elements are needed near the front and a moving coordinate system has proved effective for placing them [20]. Theoretical work [21] has shown that oscillations occur unless

$$Pe\ \Delta x < p$$

where p depends on the method. OCFE with cubics gives p = 3.46 where finite difference and Galerkin finite element (linear or quadratic elements) give p = 2. Thus small elements are needed for large Pe or the solution is degraded. This was made particularly evident by Mercer and Faust [22] in his solution of the Buckley-Leverett problem; this equation for flow in porous media is hyperbolic and OCFE performed poorly.

For one-dimensional problems the method of orthogonal collocation on finite elements has proved to be a useful, efficient method. It is especially good if high accuracy is desired, since the rate of convergence goes as h^4 or higher, where h is a typical step size. When the solution has singular-type behavior resulting in sharp fronts or large gradients, the method is not much more successful than other methods: small elements are needed.

Two-Dimensional Problems

For two-dimensional problems, the usual procedure is to take the trial function on an element as the tensor product of two one-dimensional polynomials, one in x (or ξ) and one in y (or local coordinate η). For a rectangular domain of elements collocation is applied at interior collocation points. If cubic polynomials are used in both directions, there are four collocation points corresponding to the four Gaussian quadrature points in a rectangle.

Prenter and Russell (23) have studied the convergence of the method for the elliptic equation:

$$-\frac{\partial}{\partial x}\left(p \frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(q \frac{\partial u}{\partial y}\right) + cu = f \qquad (5)$$

$$x,y\ [0, 1]$$

$$u = 0 \text{ on boundary}$$

where p, q, c, and f are functions of x and y. They expand the trial function in cubic Hermite polynomials, and prove error bounds. The error decreases as

$$\text{mean square error} \alpha h^s$$

where he is a typical dimension of an element. The parameter s takes the value 4 under some conditions ($p, q \in C^1, c \in C^0$) and is 3 otherwise. More recently Percell and Wheeler (24) have extended the proof to give errors

$$L_2 \text{ error in solution } \alpha h^{r+1}$$

for the function and

$$L_2 \text{ error in derivative } \alpha h^r$$

for the derivative. Here r is the degree of the polynomial, with cubics giving h^4.

First consider solving Eq. (5) by expanding the solution in a trial function of Hermite polynomials defined in both directions on finite elements. Evaluation of the residuals at the quadrature points gives conditions determining the coefficients of the solution. (Here first assume that the boundary conditions are easily satisfied by the trial function). Consider one element as shown in Figure 1.

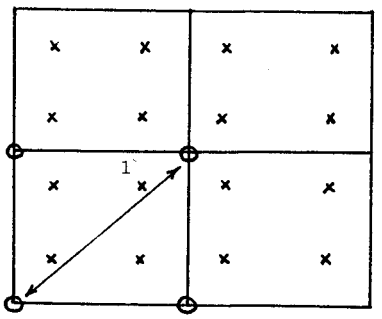

Figure 1 - Orthogonal collocation on finite elements, x- collocation point. Arrows show bandwidth. O- show parameters entering equation at point 1.

Each corner of an element has four quantities associated with it: the function, its x and y derivatives and the xy second derivative. The equation resulting from the collocation equation for point 1 then has 16 entries, since only 16 functions are needed to evaluate the differential equation at each collocation point. By contrast the Galerkin method with the same cubic Hermite trial functions has many more equations. See Figure 2.

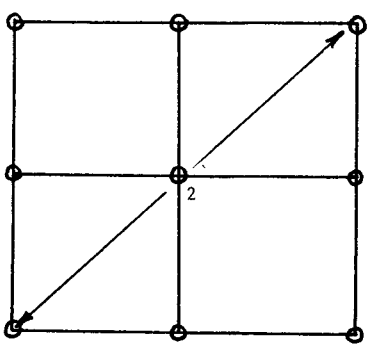

Figure 2 - Galerkin method with Hermite cubic functions. Arrows show bandwidth. O show parameters entering equation with weighting function at 2.

One of the weighting functions is a function with specific value and derivatives at point 2. The equation resulting from this weighting function in the Galerkin method will have 36 non-zero entries in it, because the weighting function overlaps four elements and all the parameters associated with each corner shown in Figure 2 must be considered. This means that the collocation method takes less work to set up the equations since the number of non-zero terms is less (16 vs. 36). Of course the extent of this advantage depends on whether the problem is linear of non-linear and whether it has a variational principle. If a variational principle exists, the matrix is symmetric and only half as many terms need be evaluated in the Galerkin method (assuming the Galerkin method is applied as a variational method when a variational principle exists).

The other advantage the collocation method has is a smaller bandwidth. The bandwidth of the matrix associated with Figure 1 is approximately $2n$, where there are n elements in one direction, and n^2 total elements. For the Galerkin method in Figure 2, the bandwidth is 4n. The following comparisons were presented at this conference two years ago.

$(1/2)\ (4n)^2(2n)^2 = 32n^4$ multiplications for Ritz method (symmetric matrix, no pivoting)

$(4n)^2(2n)^2 = 64n^4$ multiplications for Galerkin method (unsymmetric matrix, no pivoting)

$2(2n)^2(2n)^2 = 32n^4$ multiplications for collocation method (unsymmetric matrix with pivoting)

For the decomposition cost collocation is equivalent to the Ritz method and is twice as fast as the Galerkin method.

The above analysis is changed when we must include the nodes and parameters associated with boundaries. This may be necessary because a program is to be written to allow many different kinds of boundary conditions, which the analyst chooses, making the automatic renumbering of the unknown difficult to program for arbitrary cases. When all unknowns on the boundary are included, the collocation bandwidth increases to 4n whereas the Galerkin and Ritz bandwidths remain the same. Then the relative comparisons for LU decomposition are $32n^4$ for Ritz, $64n^4$ for Galerkin, and $128n^4$ for collocation. By skipping the multiplication of rows by zeroes in the LU decomposition, economies can be made. Houstis [25] reports that the decomposition time for collocation is between that of the Galerkin and Ritz methods, when the zeroes are taken into account. Thus in practice the LU decomposition time for collocation is slightly smaller than that for Galerkin when both use cubic polynomials.

The total comparison of collocation with Galerkin, both methods using cubic polynomials, depends on the problem. If the problem is linear, then the LU decomposition time predominates and collocation is only slightly better than Galerkin. If the problem is non-linear, then the setup time is significant and then the collocation method is much preferred. We see below the things given up (at least until now) by the choice of collocation.

The above comparisons are for direct solution methods, i.e., solving the algebraic equations as a system. Iterative methods are also attractive. Two years ago at this conference an alternating-direction method was presented for collocation on finite elements [26]. The method used Lagrangian polynomials and solved the equation first along one row in x, then the next row in x, etc., until the entire domain was swept. Then another sweep was made on the y rows. Details were given on the spectral radius of the matrices that arise. Briefly it was found that spectral radius increased (and so did the number of iterations required) whenever the number of elements or the degree of polynomial increased. For rectangular arrays with all the elements the same size, the ADI method proved to be from 4 to 6 times faster than the direct solution. If there were both small and large elements, however, the spectral radius increased and the direct methods proved superior. This beginning study emphasizes the need for more development of iterative methods, such as the Laplace modified methods being worked on by Professor Wheeler. The matrix arising in collocation is so sparse, but with a regular structure, that an iterative procedure than can capitalize on the sparsity will prove very powerful.

Applications

At this conference two years ago, Houstis and Rice [27] presented collocation software for the solution of linear elliptic problems on general two-dimensional domains,

$$\alpha\frac{\partial^2 u}{\partial x^2} + 2\beta\frac{\partial^2 u}{\partial x \partial y} + \gamma\frac{\partial^2 u}{\partial y^2} + \delta\frac{\partial u}{\partial x} + \varepsilon\frac{\partial u}{\partial y} + \zeta\, u = f$$

The boundary conditions on the irregular domain had to be treated very carefully to insure good results. A detailed comparison of finite element collocation to finite difference and Galerkin finite element methods was made by Houstis, et al. [25]. The problems chosen for comparison were linear elliptic boundary value problems, sometimes with irregular domains. All finite element methods used Hermite bicubics. The collocation finite element method proved superior to Galerkin and Least Squares finite element methods, and was usually superior to the finite difference method. For good accuracy, the collocation method always was more efficient than finite difference. This is the only careful, controlled comparison of methods for two-dimensional problems, and is limited to linear problems.

Recently the author has had the opportunity to compare the performance of the collocation finite element method to that of the Galerkin finite element method for an important class of flow problems. This comparison illustrates the considerations in "real-life" problems. The problems involved the slow, steady flow of fluid in two dimensions. For a Newtonian fluid the equation is

$$0 = -\nabla p + \mu\nabla^2\underline{u}$$

whereas for a purely viscous fluid we must solve

$$0 = -\nabla p + \nabla \cdot (1/2\eta(|\nabla u|)(\nabla\underline{u} + \nabla\underline{u}^T))$$

and for a viscoelastic fluid the equation is much more complicated. In addition the continuity equation is

$$\nabla \cdot \underline{u} = 0.$$

The Galerkin program used a frontal or profile solver, whereas the collocation method used a block diagonal LU decomposition. Detailed comparisons are given elsewhere [28], but let us see the major results.

The problems solved are illustrated in Figure 3.

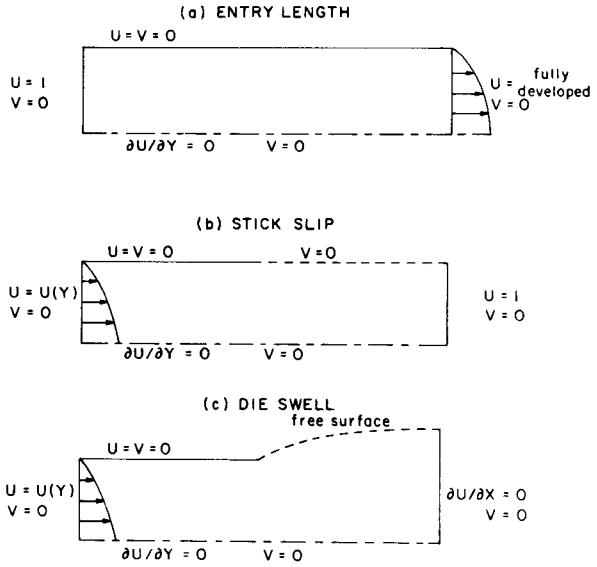

Figure 3 - Flow geometries studied.

In the entry flow, the velocity is 1 at the left hand side and 0 on the top boundary. Thus the exact solution is discontinuous at the corner. In the stick slip problem the fluid is zero along the top surface until the edge of the die lip, and then it accelerates along the top free surface. In this case, the first derivative of velocity is discontinuous and infinite at the die lip. In the die swell the shape of the jet is allowed to change until the normal and tangential forces are balanced. A singularity also occurs for this problem.

At the outset the Galerkin method may look better: an element only has 22 unknowns whereas a collocation element has 48. Thus the collocation method must use many fewer elements to be competitive. However, because of the problems chosen for study the high order accuracy of the collocation method cannot be realized because the rate of convergence is governed by the continuity of the exact solution rather than the degree of polynomial. This does not augur well for the collocation method. In fact it was found that for the Newtonian fluid the collocation method did as well as the Galerkin method. The collocation method with a few elements gave better integral mass and force balances than did the Galerkin method with more elements. The Galerkin method, which could use mesh refinement because it had more elements, gave more accurate local properties of the solution. When the fluids were non-Newtonian, and the viscosity depended on the shear rate, the collocation methods were not competitive. In addition, the collocation method was much more sensitive to the iteration scheme on viscosity than the Galerkin method. When viscoelastic fluids were used, the collocation methods proved to be very expensive. In addition the Galerkin method had the capability of refining the mesh in arbitrary ways since it used isoparametric elements. The collocation method treated irregular domains by transforming the problem to a regular domain: thus the range of possibilities was much less. Based on all these considerations, in the subsequent work it was decided to use the Galerkin method because of its greater versatility in treating irregular domains, its ability to use refined meshes, and a desire to solve flow problems which contained singularities so that the high accuracy of high-order methods could not be achieved.

Another potential area of application is the flow of fluids through porous media. Petroleum companies solve these nonlinear elliptic/parabolic problems to predict performance of oil reservoirs. The finite difference methods are finely tuned: several iterative methods are very efficient [29]. Galerkin finite element methods have been used, but usually at a great penalty in computer cost [30]. More recently, Galerkin methods using iterative methods have proved more competitive [31, 32]. It is clear, however, that the straightforward application of Galerkin finite element methods to petroleum reservoir problems does not give competitive results. Advances need to be made in iterative solution techniques to become competitive. The same comment must hold for collocation finite element methods.

Future Development

The collocation finite element method is being developed because of its promise of reduced computational cost. Widespread use will not happen unless the following problems are addressed. For many engineering problems, it is necessary to solve problems on irregular domains. Thus the isoparametric elements used in Galerkin methods are very useful. Pinder, et al. [33] have developed isoparametric transformations for use with collocation, and these would permit treatment of arbitrary domains. Another advance would be the C^0 collocation-weighted residual methods in two dimensions [24]. Once the constraints are dropped for continuity of derivatives, the deformation of elements becomes very much easier. Mesh refinement is an important feature of finite element methods and is easily done in Galerkin methods with triangular elements. Collocation on triangles would be useful if such a method were developed. Methods have been advanced [34, 31] which are closely related to collocation methods but which have some efficiencies or ability to treat irregular geometries. One discouraging feature of high-order methods, collocation or Galerkin, is that problems which may be well behaved in one dimension exhibit singularities easily in two dimensions. Then the high-order methods do not achieve high accuracy and the approximation is made better only by adding elements and refining the mesh. But the high-order elements contain so many unknowns that the problems become very big very easily. Then the low-order methods prove attractive.

Conclusion

As the collocation field matures, better iterative methods will be developed, irregular domains will be treated, and maybe even collocation on triangles will prove possible. Despite these possible successes, it should be realized that there are some problems that are stacked against a high-order, good accuracy collocation method. There are enough other problems, however, that progress is worthwhile.

References

1. Finlayson, B. A., "The Method of Weighted Residuals and Variational Principles," Academic Press (1972).
2. Sincovec, R. F., Math. Comp. 26 893 (1972).
3. deBoor, C., J. Approx. Theory 6 50 (1972).
4. Carey, G. F. and Finlayson, B. A., Chem. Eng. Sci. 30 587 (1975).
5. Wheeler, M. F., SIAM J. Numer. Anal. 14 71 (1977).
6. Carey, G. F., Humphrey, D. and Wheeler, M. F., "Finite Element Methods for Catalyst Pellet Models," to be published.
7. Hindmarsch, A. C., Report UCID - 30059, Rev. 1, Lawrence Livermore Laboratory (March, 1975).
8. Madsen, N. K. and Sincovec, R. F., "PDECOL:General Collocation Software for Partial Differential Equations," UCRL-78263, May, 1976, to appear in ACM Trans. Math. Software.
9. Douglas, J., Jr., pp. 475-490 in The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations, A. K. Aziz (ed.), Academic Press (1972).
10. Douglas, J.,Jr. and Du Pont, T., Math. Comp. 27 17 (1973).
11. deBoor, C. and Swartz, B., SIAM J. Numer. Anal. 10 582 (1973).
12. Hulme, B. L., Math. Comp. 26 415 (1972).
13. Dunn, R. J., Jr. and Wheeler, M. F., SIAM J. Numer. Anal. 13 720 (1976).
14. Chawla, T. C., Leaf, G., Chen, W. L., and Grolmes, M. A., Trans. ASME 97, Ser. C., J. Heat Trans. 562 (1975).
15. Finlayson, B. A., pp. 3.91-3.106 in "Finite Elements in Water Resources," Gray, W. G., Pinder, G. F., and Brebbia, C. A. (ed.), Pentech (1977).
16. Young, L. C. and Finlayson, B. A., Am. Inst. Chem. Eng. J. 22 331, 343 (1976); 25 192 (1979).
17. Lee, S. T. and Aris, R., Chem. Eng. Sci. 32 827 (1977).
18. Birnbaum, I. and Lapidus, L., Chem. Eng. Sci. 33 443 (1978).
19. Hopkins, T. R. and Wait, R., Int. J. Num. Methods Eng. 12 1081 (1978).
20. Jensen, O. K. and Finlayson, B. A., pp. 4.21-4.32 in "Finite Elements in Water Resources," Brebbia, C. A., Gray, W. G. and Pinder, G. F. (ed.), Pentech Press (1978).
21. Jensen, O. K. and Finlayson, B. A., "Oscillation Limits for Weighted Residual Methods," to be published.
22. Mercer, J. W. and Faust, C. R., pp. 1.21-1.58 in "Finite Elements in Water Resources," Gray, W. G., Pinder, G. F., and Brebbia, C. A. (ed.), Pentech (1977).
23. Prenter, P. M. and Russell, R. D., SIAM J. Numer. Anal. 13 923 (1976).
24. Percell, P. and Wheeler, M. F., private communication (1979).
25. Houstis, E. N., Lynch, R. E., Rice, J. R., and Papatheodorou, T. S., J. Comp. Phys. 27 323 (1978).
26. Chang, P. W. and Finlayson, B. A., Math. Comp. Sim. 20 83 (1978).
27. Houstis, E. N. and Rice, J. R., pp. 7-12 in Adv. Comp. Methods for Partial Differential Equations-II, R. Vichnevetsky (ed.), IMACS (1977).
28. Chang, P. W., Patten, T. W., and Finlayson, B. A., Comp. Fluids, to be published.
29. Peaceman, D. W., "Fundamentals of Numerical Reservoir Simulation," Elsevier (1977).
30. Spivak, A., Price, H. S., and Settari, A., SPE paper 5723, pp. 69-91 in "Proceedings of Symposium on Numerical Simulation of Reservoir Performance," Soc. Pet. Eng., Dallas, Texas (1976).
31. Young, L. C., pp. 307-320 in Adv. Comp. Methods for Partial Differential Equations-II, R. Vichnevetsky (ed.), IMACS (1977).
32. Young, L. C., Soc. Pet. Eng. paper 7413, Houston, Tex., Oct. 1-3, 1978.
33. Pinder, G. F., Frind, E. O., Celia, M. A., pp. 1.171-1.185 in "Finite Elements in Water Resources," Brebbia, C. A., Gray, W. G. and Pinder, G. F. (ed.), Pentech Press (1978).
34. Gray, W. G., pp. 4.33-4.50 in "Finite Elements in Water Resources," Gray, W. G., Pinder, G. F., and Brebbia, C. A. (ed.), Pentech (1977).