

## Convective instability of ferromagnetic fluids

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Convective instability of a ferromagnetic fluid is predicted for a fluid layer heated from below in the presence of a uniform vertical magnetic field. Convection is caused by a spatial variation in magnetization which is induced when the magnetization of the fluid is a function of temperature and a temperature gradient is established across the layer. A linearized convective instability analysis predicts the critical temperature gradient when only the magnetic mechanism is important, as well as when both the magnetic and buoyancy mechanisms are operative. The magnetic mechanism predominates over the buoyancy mechanism in fluid layers about 1 mm thick. For a fluid layer contained between two free boundaries which are constrained flat, the exact solution is derived for some parameter values and oscillatory instability cannot occur. For rigid boundaries, approximate solutions for stationary instability are derived by the Galerkin method for a wide range of parameter values. It is shown that in this case the Galerkin method yields an eigenvalue which is stationary to small changes in the trial functions, because the Galerkin method is equivalent to an adjoint variational principle.

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### 1. Introduction

Thermo-mechanical interactions in fluids make possible convection induced by externally applied temperature gradients. The most familiar example of thermo-mechanical interaction is buoyancy-induced convection, in which case the seat of the driving force resides in the body force, gravity, and the density is a function of temperature. Convection can also be induced by thermo-mechanical interactions arising through the stress tensor, such as surface-tension-driven convection, provided the surface tension is a function of temperature. While this mechanism occurs only at a free surface, a thermo-mechanical interaction arising through the stress tensor and acting throughout the fluid was predicted by Finlayson & Scriven (1969) after making formal postulates about the constitutive relation for the stress tensor and its dependence on temperature gradients (and concentration gradients). Convective instability analyses are useful in all three cases to predict the critical temperature gradient above which motion occurs. We study here a thermo-mechanical interaction which is predicted for a ferromagnetic fluid in the presence of a uniform, vertical magnetic field provided the magnetization is a function of temperature and a temperature gradient is established across a fluid layer.

The mechanism is similar to that predicted by Poots (1963), Yeung & Yu (1968) and Turnbull (1969), for a heated, dielectric fluid in the presence of an electric field. Only the last author, however, allowed the electric field to be influenced by the motion. Turnbull solved for the combined electric and buoyancy mechanisms and assumed the dielectric constant depended only on temperature, except that dependence on electric field was allowed in the quiescent state. Here we solve for the combined magnetic and buoyancy mechanisms, as well as the magnetic mechanism alone (applicable to thin fluid layers), and allow an equation of state which permits the magnetization to depend on both temperature and magnetic field.

Ferromagnetic fluids are formed by suspending submicron sized particles of magnetite in a carrier medium such as kerosene, heptane or water (Papell & Faber 1966, 1968; Cowley & Rosensweig 1967; Rosensweig, private communication). To prevent the particles from agglomerating in the presence of a magnetic field they are surrounded by a surfactant such as oleic acid. The combination of the short range repulsion due to the surfactant and the thermal agitation yields a material which behaves as a continuum (Papell & Faber 1966) and can experience forces due to magnetic polarization. The fluids are usually good insulators and forces due to interaction of magnetic fields with currents of free charge, such as found in magnetohydrodynamics, are negligible (Cowley & Rosensweig 1967). The presence of a ferromagnetic fluid can distort an external magnetic field if magnetic interaction (dipole-dipole) takes place, but this is negligible for small particle concentrations (Bean 1955), as is assumed here. Experience also suggests that hysteresis is unlikely in the fluids (Cowley & Rosensweig 1967), except for rapidly changing external magnetic fields (see Moskowitz & Rosensweig 1967), and here we assume that the magnetic field  $\mathbf{H}$  and magnetic induction  $\mathbf{B}$  are parallel.

We study the convective instability of a ferromagnetic fluid when placed in a fluid layer which is heated from below when in the presence of a vertical magnetic field. We analyze the instability both in the presence and absence of a vertical body force (gravity). When buoyancy forces are negligible, as is the case for thin fluid layers, the external magnetic field induces a magnetization in the ferromagnetic fluid. In the magnetic equation of state the magnetization is a function of both magnetic field and temperature, so that the applied temperature gradient causes a spatial variation in the magnetization, which is the driving force causing the convection. For small temperature gradients the magnetic forces induced by the temperature gradient cannot overcome the viscous and thermal dissipation and the layer remains quiescent. When the temperature gradient is sufficiently large, motion occurs. When both buoyancy and magnetic forces must be included, the magnetic mechanism changes the critical Rayleigh number applicable to natural convection. The analysis also yields an interesting mathematical result presented in §4.2: for this problem the Galerkin method yields an eigenvalue which is stationary to small changes in the trial functions, a feature usually associated only with variational principles. Here the result depends upon the equivalence between the Galerkin method and an adjoint variational method whenever the adjoint boundary conditions are the same as

the original boundary conditions, as is the case in a wide variety of convective instability problems.

## 2. Derivation of equations

The momentum equation for an incompressible fluid with constant viscosity is

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p' + \rho \mathbf{g} + \nabla \cdot (\mathbf{H}\mathbf{B}) + \mu \nabla^2 \mathbf{u}, \quad (1)$$

where  $\rho$  is the density,  $\mathbf{u}$  the velocity,  $t$  is time,  $p'$  the pressure (the magnetic contribution to pressure is discussed in the appendix),  $\mathbf{g}$  the gravitational body force,  $\mu$  is the viscosity,  $\mathbf{H}$  the magnetic field and  $\mathbf{B}$  the magnetic induction. The additional term pertinent to a ferromagnetic fluid is the magnetic stress, which is derived by Landau & Lifshitz (1960), Cowley & Rosensweig (1967) and Penfield & Haus (1967). Two additional complications are assumed negligible in the above equation: we assume the viscosity is isotropic and independent of the magnetic field. Both approximations simplify the analysis without changing the ultimate conclusion. We also employ the Boussinesq approximation by allowing the density to change only in the gravitational body force term.

The temperature equation for an incompressible fluid which obeys Fourier's law is (see the appendix)

$$\left[ \rho C_{VH} - \mu_0 \mathbf{H} \cdot \left( \frac{\partial \mathbf{M}}{\partial T} \right)_{V, \mathbf{H}} \right] \frac{dT}{dt} + \mu_0 T \left( \frac{\partial \mathbf{M}}{\partial T} \right)_{V, \mathbf{H}} \cdot \frac{d\mathbf{H}}{dt} = k_1 \nabla^2 T + \Phi, \quad (2)$$

where  $C_{VH}$  is the heat capacity at constant volume and magnetic field,  $T$  is temperature,  $\mathbf{M}$  is the magnetization (defined by (4) below),  $k_1$  is the thermal conductivity (assumed constant), and  $\Phi$  is the viscous dissipation. The partial derivatives of  $\mathbf{M}$  are material properties which can be evaluated once the magnetic equation of state, such as (6) below, is known.

Maxwell's equations, simplified for a non-conducting fluid with no displacement currents, become

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = 0. \quad (3a, b)$$

In the Chu formulation of electrodynamics (Penfield & Haus 1967), the magnetization,  $\mathbf{M}$ , and magnetic field,  $\mathbf{H}$ , are used as primary quantities rather than  $\mathbf{B}$  and  $\mathbf{H}$ . The three are related by

$$\mathbf{B} \equiv \mu_0 (\mathbf{H} + \mathbf{M}). \quad (4)$$

We assume that the magnetization is aligned with the magnetic field, but allow a dependence on the magnitude of the magnetic field as well as the temperature

$$\mathbf{M} = \frac{\mathbf{H}}{H} M(H, T). \quad (5)$$

The magnetic equation of state is linearized about the magnetic field,  $H_0$ , and an average temperature,  $T_a$ , to become

$$M = M_0 + \chi(H - H_0) - K(T - T_a), \quad (6)$$

where the susceptibility and the pyromagnetic coefficient are defined

$$\chi \equiv \left( \frac{\partial M}{\partial H} \right)_{H_0, T_a}, \quad K \equiv - \left( \frac{\partial M}{\partial T} \right)_{H_0, T_a}. \quad (7)$$

$H_0$  is the uniform magnetic field of the fluid layer when placed in an external magnetic field  $\mathbf{H} = \mathbf{k}H_0^{\text{ext}}$ . Thus the analysis is restricted to physical situations in which the magnetization induced by temperature variations is small compared to that induced by the external magnetic field. The density equation of state is taken as

$$\rho = \rho_0(1 - \alpha(T - T_a)), \quad (8)$$

where the constant  $\alpha$  is the thermal expansion coefficient.

The magnetic boundary conditions are that the normal component of magnetic induction and tangential component of magnetic field are continuous across the boundary. The usual velocity boundary condition is  $\mathbf{u} = 0$  on a rigid wall and the temperature is assumed constant on each boundary.

$$T = T_0 \quad \text{at} \quad z = \frac{1}{2}d, \quad T = T_1 \quad \text{at} \quad z = -\frac{1}{2}d, \quad T_a = \frac{1}{2}(T_0 + T_1),$$

where  $d$  is the thickness of the fluid layer.

In the quiescent state, the solution of (1)–(8) is

$$\mathbf{u} = 0, \quad T = T_a - \beta z, \quad \beta = \frac{T_1 - T_0}{d}, \quad (9)$$

$$\mathbf{H}_0 = \mathbf{k} \left( H_0 - \frac{K\beta z}{1 + \chi} \right), \quad \mathbf{M}_0 = \mathbf{k} \left( M_0 + \frac{K\beta z}{1 + \chi} \right), \quad H_0 + M_0 = H_0^{\text{ext}}. \quad (10)$$

Only the spatially varying parts of  $\mathbf{H}_0$  and  $\mathbf{M}_0$  contribute to the analysis, so that the direction of the external magnetic field is unimportant and the convective phenomenon is the same whether the external magnetic field is parallel or antiparallel to the gravitational force.

We next study the stability of this quiescent state with a linearized analysis. Equations (5) and (6) yield

$$\left. \begin{aligned} H'_3 + M'_3 &= (1 + \chi) H'_3 - K T', \\ H'_i + M'_i &= (1 + M_0/H_0) H'_i \quad (i = 1, 2), \end{aligned} \right\} \quad (11)$$

where we have assumed  $K\beta d \ll (1 + \chi)H_0$ . Equation (3b) means we can write  $\mathbf{H}' = \nabla\Phi'$ . The vertical component of the vorticity equation is

$$\rho \frac{\partial Z}{\partial t} = \mu \nabla^2 Z, \quad (12)$$

where  $Z = \mathbf{k} \cdot (\nabla \times \mathbf{u})$ . Since  $Z$  must vanish on the boundary, (12) predicts that any perturbation in vorticity must decay in time and in the instability analysis we set  $Z = 0$  without loss of generality. As is customary in convective instability analyses we assume the normal mode hypothesis or separation of variables. Each variable is expanded in the form

$$T'(x, y, z, t) = T(z, t) \exp(ik_x x + ik_y y).$$

The vertical component of the curl of the vorticity equation is

$$-\rho \frac{\partial}{\partial t} (D^2 - k^2)W = +k^2 \alpha \rho g T - \frac{\mu_0 K \beta}{1 + \chi} [(\chi + 1) D\Phi - KT] k^2 - \mu (D^2 - k^2)^2 W; \quad (13)$$

the temperature equation is

$$\rho C \frac{\partial T}{\partial t} - \mu_0 T K \frac{\partial D\Phi}{\partial t} = k_1 \nabla^2 T + \left( \rho C \beta - \frac{\mu_0 T_0 K^2 \beta}{1 + \chi} \right) W, \quad (14)$$

where  $\rho C = \rho C_{VH} + \mu_0 K H_0$ , and (3a) becomes

$$(1 + \chi) D^2 \Phi - (1 + M_0/H_0) k^2 \Phi - KDT = 0. \quad (15)$$

The boundary conditions on velocity and temperature are

$$W = DW = T = 0 \quad \text{at} \quad z = \pm \frac{1}{2}d. \quad (16)$$

The boundary conditions on the magnetic potential,  $\Phi$ , are complicated by the fact that the periodic nature of  $\Phi$  within the fluid layer induces a periodic magnetic potential outside the layer. Thus outside the layer the magnetic potential is governed by

$$(D^2 - k^2) \Psi = 0.$$

This equation can be solved subject to (3), which takes the form,

$$\Psi = \Phi, \quad D\Psi = (1 + \chi) D\Phi - KT \quad \text{at} \quad z = \pm \frac{1}{2}d$$

to obtain the boundary conditions on  $\Phi$ :

$$\left. \begin{aligned} (1 + \chi) D\Phi - k\Phi &= 0 \quad \text{at} \quad z = \frac{1}{2}d, \\ (1 + \chi) D\Phi + k\Phi &= 0 \quad \text{at} \quad z = -\frac{1}{2}d. \end{aligned} \right\} \quad (17)$$

We next put (13)–(15) in dimensionless form by using the standards:  $W_s = \nu/d$ ,  $T_s = \rho C \beta \nu d / (k_1 a R^{\frac{1}{2}})$ ,  $\Phi_s = KT_s d / (1 + \chi)$ ,  $z_s = d$ ,  $t_s = \rho d^2 / \mu$ . The final equations are

$$\left. \begin{aligned} -\frac{\partial}{\partial t} (D^2 - a^2)W &= -(D^2 - a^2)^2 W + aR^{\frac{1}{2}}[(1 + M_1)T - M_1 D\Phi], \\ P \frac{\partial T}{\partial t} - PM_2 \frac{\partial D\Phi}{\partial t} &= (D^2 - a^2)T + aR^{\frac{1}{2}}(1 - M_2)W, \\ 0 &= D^2 \Phi - M_3 a^2 \Phi - DT, \end{aligned} \right\} \quad (18a, b, c)$$

where

$$P = \frac{\mu C}{k_1}, \quad M_1 = \frac{\mu_0 K^2 \beta}{(1 + \chi) \alpha \rho g}, \quad M_2 = \frac{\mu_0 T_0 K^2}{\rho C (1 + \chi)}, \quad M_3 = \frac{1 + M_0/H_0}{1 + \chi},$$

$$R = \frac{\alpha g \beta d^4 \rho C}{\nu k_1}, \quad N = RM_1 = \frac{\mu_0 K^2 \beta^2 d^4 \rho C}{\mu k_1 (1 + \chi)},$$

and  $a$  is the dimensionless wave-number. This set of equations must be solved subject to the boundary conditions (16) and (17) and the system reduces to an eigenvalue problem for  $R$ . For the special case when buoyancy forces are negligible, the analogous equations can be obtained from (18) by replacing  $R$  by  $N$ , and in (18a) setting  $(1 + M_1)T = T$ ,  $M_1 D\Phi = D\Phi$ . Then the temperature

perturbation is measured in units of  $T_s = \rho C \beta v d / (k_1 a N^{\frac{1}{2}})$ , and the system reduces to an eigenvalue problem in  $N$ . We note that the parameter  $M_1$  is a ratio of the magnetic to gravitational forces. The parameter  $M_3$  measures the departure of linearity in the magnetic equation of state and values from one ( $M_0 = \chi H_0$ ) to higher values are possible for the usual equations of state.

### 3. Exact solution for free boundaries

We first consider the problem when the boundary conditions on velocity are changed to those appropriate to a free surface which is constrained flat. While this case is of little physical interest, it is mathematically important because we can derive an exact solution whose properties guide our analysis below. We consider here only the case in which  $\chi \rightarrow \infty$ .

The boundary conditions are†

$$W = D^2W = T = D\Phi = 0 \quad \text{at} \quad z = \pm \frac{1}{2}.$$

The solution can be separated into even and odd modes and we expect the even modes give the lowest eigenvalue. Consequently, we consider solutions in which  $W$ ,  $T$  and  $D\Phi$  are even, but  $\Phi$  is odd and symmetric about  $z = 0$ . The exact solution is then

$$\begin{aligned} W &= A e^{\sigma t} \cos \pi z, & T &= B e^{\sigma t} \cos \pi z, \\ D\Phi &= C e^{\sigma t} \cos \pi z, & \Phi &= + (C/\pi) e^{\sigma t} \sin \pi z. \end{aligned}$$

These functions are substituted into the set of equations (18), which can be satisfied provided the constants  $A$ ,  $B$  and  $C$  are chosen appropriately. We thus obtain a set of three linear, homogeneous algebraic equations in the constants  $A$ ,  $B$  and  $C$ . A solution exists if and only if the determinant of the coefficients vanishes, leading to the characteristic equation

$$U\sigma^2 + V\sigma + W = 0,$$

where

$$\left. \begin{aligned} U &= P(\pi^2 + a^2) [\pi^2(1 - M_2) + M_3 a^2], \\ V &= (\pi^2 + a^2)^2 [(1 + P)(\pi^2 + M_3 a^2) - P M_2 \pi^2], \\ W &= (\pi^2 + a^2)^3 (\pi^2 + M_3 a^2) - a^2 R (1 - M_2) [(1 + M_1)(\pi^2 + M_3 a^2) - \pi^2 M_1]. \end{aligned} \right\} \quad (19)$$

This equation determines the eigenvalue  $R$  for which solutions exist. If neutral oscillatory instability occurs the time factor  $\sigma = i\omega$ . Since the functions  $U$ ,  $V$  and  $W$  are all real, (19) can be satisfied for  $\sigma = i\omega$ , only if  $V = 0$ . Typical values of  $M_2$  are  $+10^{-6}$  so that  $V$  is positive and oscillatory instability cannot occur. For stationary instability (and  $M_2 \simeq 0$ ) the Rayleigh number is given by

$$R = \frac{(\pi^2 + a^2)^3}{a^2 [1 + M_1 - M_1 (\pi^2 / [\pi^2 + M_3 a^2])]}, \quad (20)$$

† The free stress boundary condition must be generalized to include the Maxwell stress terms  $\mathbf{HB}$ . However, because of the boundary conditions applicable to  $\mathbf{H}$  and  $\mathbf{B}$ , the additional terms drop out, leaving only  $D^2W = 0$ .

which must be minimized with respect to the wave-number to find the critical Rayleigh number.

When  $M_1 = 0$  we get the classical Rayleigh problem for buoyancy-induced convection with  $a_c^2 = \frac{1}{2}\pi^2$  and  $R_c = \frac{27}{4}\pi^4$ . For  $M_1$  very large, we obtain the results for the magnetic mechanism operating in the absence of buoyancy effects.

$$N = RM_1 = \frac{(\pi^2 + a^2)^3 (\pi^2 + M_3 a^2)}{M_3 a^4}. \quad (21)$$

The critical wave-number and magnetic number,  $N_c$ , depend on the parameter  $M_3$ , taking the values

$$N_c = 16\pi^4, \quad a_c^2 = \pi^2 \quad \text{for} \quad M_3 = 1,$$

and

$$N_c = \frac{27}{4}\pi^4, \quad a_c^2 = \frac{1}{2}\pi^2 \quad \text{for} \quad M_3 \rightarrow \infty$$

and intermediate values for intermediate  $M_3$ . If we write  $N_c = K\pi^4$ , where  $K$  depends on  $M_3$ , we can rearrange (20) to demonstrate the interaction of the buoyancy and magnetic modes of instability.

$$\frac{R}{R_c} + \frac{N}{N_c} \frac{4K}{27} \left[ \frac{M_3 X^2}{1 + M_3 X^2} \right] = \frac{4}{27} \frac{(1 + X^2)^3}{X^2}, \quad (22)$$

where  $X = a/\pi$ . When  $M_3$  is very large (21) reduces to

$$\frac{R}{R_c} + \frac{N}{N_c} = 1.$$

This result demonstrates a tight coupling between the buoyancy and magnetic forces which is possible because each individual convective mechanism yields the same wave-number. Such tight coupling is reminiscent of that obtained in the combined buoyancy and surface tension instability discussed by Nield (1964). It holds here only for special values of the parameters; otherwise (22) must be used. Numerical results concerning the coupling are deferred until treating two rigid boundaries, where the numerical results are of more interest physically.

## 4. Solution for rigid boundaries

### 4.1. Galerkin method

For rigid boundaries and a finite  $\chi$  we must consider oscillatory instability. The first approximation of the Galerkin method, however, if applied in the manner described by Finlayson (1968), yields a result similar to (19), suggesting that oscillatory instability does not occur even for the more general boundary conditions. Thus we limit consideration to stationary instability.

The set of equations (18), simplified for stationary instability and with  $M_2 = 0$ , can be conveniently represented in matrix notation

$$\mathbf{L} \cdot \mathbf{W} = aR^{\frac{1}{2}} \mathbf{M} \cdot \mathbf{W}, \quad (23)$$

where

$$\mathbf{W} = \{W, T, \Phi\},$$

$$\mathbf{L} = \begin{Bmatrix} (D^2 - a^2)^2 & 0 & 0 \\ 0 & -(D^2 - a^2) & 0 \\ 0 & D & -(D^2 - a^2 M_3) \end{Bmatrix},$$

$$\mathbf{M} = \begin{Bmatrix} 0 & 1 + M_1 & -M_1 D \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix}.$$

The set of equations is invariant under the transformation  $z \rightarrow -z$ ,  $W \rightarrow W$ ,  $T \rightarrow T$ ,  $\Phi \rightarrow -\Phi$ , so that the exact solution, represented as a power series, contains only even powers of  $z$  for velocity and temperature and only odd powers of  $z$  for the magnetic potential. Numerical values of the eigenvalue are obtained using the Galerkin method by expanding the velocity, temperature and magnetic potential in the series

$$W = \sum_{i=1}^N A_i (z^2 - \frac{1}{4})^{i+1}, \quad T = \sum_{i=1}^N B_i (z^2 - \frac{1}{4})^i, \quad \Phi = \sum_{i=1}^{N+2} C_i z^{2i-1}.$$

The velocity and temperature trial functions satisfy the boundary conditions, (16), whereas the magnetic potential does not satisfy its boundary conditions (17). These functions are substituted into (23) to obtain the residuals. The velocity equation is required to be orthogonal to each velocity trial function  $(z^2 - \frac{1}{4})^{j+1}$ , and the temperature equation is made orthogonal to each temperature trial function. For the magnetic potential we must include boundary residuals in the following way. Define the differential equation and boundary residuals as

$$R_D = (D^2 - a^2 M_3) \Phi - DT,$$

$$R_B(\frac{1}{2}) = - \left( D\Phi + \frac{a}{1+\chi} \Phi \right) \Big|_{z=\frac{1}{2}},$$

$$R_B(-\frac{1}{2}) = + \left( D\Phi - \frac{a}{1+\chi} \Phi \right) \Big|_{z=-\frac{1}{2}}.$$

The combined inner product is set to zero

$$\langle \Phi_j R_D \rangle + \Phi_j(\frac{1}{2}) R_B(\frac{1}{2}) + \Phi_j(-\frac{1}{2}) R_B(-\frac{1}{2}) = 0,$$

where

$$\langle uv \rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} uv dz.$$

Upon integrating the first term by parts we find that the part of the boundary residual involving derivatives is cancelled. This corresponds exactly to the use of natural boundary conditions in the calculus of variations. Such a technique for the Galerkin method is not widely recognized, although it has been used before (Bolotin 1963; Mikhlin 1964). If these manipulations are carried out, we obtain a set of homogeneous linear equations in the expansion coefficients of the trial functions. This set of equations has a solution if and only if the determinant of the coefficients vanishes, which leads to a characteristic equation to be solved for the eigenvalue,  $R$ .

4.2. Adjoint variational method

Before discussing the numerical results we show that the Galerkin method is equivalent, in this case, to an adjoint variational principle. Thus the eigenvalue is stationary. The adjoint operator is defined by requiring that

$$\langle \mathbf{W}^* \cdot (\mathbf{L} - aR^{\frac{1}{2}}\mathbf{M}) \cdot \mathbf{W} - \mathbf{W} \cdot (\mathbf{L}^* - aR^{\frac{1}{2}}\mathbf{M}^*) \cdot \mathbf{W}^* \rangle = \text{boundary terms} = 0$$

and the result is 
$$\mathbf{L}^* \cdot \mathbf{W}^* = aR^{\frac{1}{2}}\mathbf{M}^* \cdot \mathbf{W}^*, \tag{24}$$

where 
$$\mathbf{L}^* = \left\{ \begin{array}{ccc} (D^2 - a^2)^2 & 0 & 0 \\ 0 & -(D^2 - a^2) & -D \\ 0 & 0 & -(D^2 - a^2M_3) \end{array} \right\},$$

$$\mathbf{M}^* = \left\{ \begin{array}{ccc} 0 & 1 & 0 \\ 1 + M_1 & 0 & 0 \\ M_1 D & 0 & 0 \end{array} \right\}.$$

The adjoint boundary conditions are the same as the original boundary conditions. An adjoint variational principle can be derived in the manner suggested by Roberts (1960) and Chandrasekhar (1961). The functional

$$aR^{\frac{1}{2}} = \frac{\langle \mathbf{W}^* \cdot \mathbf{L} \cdot \mathbf{W} \rangle}{\langle \mathbf{W}^* \cdot \mathbf{M} \cdot \mathbf{W} \rangle}$$

is to be made stationary among all possible variations of the trial functions  $\mathbf{W}$  and  $\mathbf{W}^*$ . It is easily seen that the Euler equations are just (23) and (24).

We next consider the application of this adjoint variational principle, but use a single equation rather than a system of equations for simplicity. Then

$$\lambda \equiv - \frac{\langle u^*, Lu \rangle}{\langle u^*, Mu \rangle}.$$

We expand the functions  $u$  and  $u^*$  in the series

$$u = \sum c_i u_i, \quad u^* = \sum c_i^* u_i^*.$$

Application of the variational principle yields

$$\left. \begin{array}{l} \partial\lambda/\partial c_k^* = 0: \sum_j c_j \langle u_k^*, Lu_j + \lambda Mu_j \rangle = 0; \\ \partial\lambda/\partial c_k = 0: \sum_j c_j^* \langle u_k, L^*u_j^* + \lambda M^*u_j^* \rangle = 0. \end{array} \right\} \tag{25}$$

Yet the matrices in these equations are transposes of each other, after we use the property of the operators  $\langle u_k^*, Lu_j \rangle = \langle u_j, L^*u_k^* \rangle$  and similarly for  $M$ . Furthermore, when the first set of equations has a non-trivial solution, so does the second set. Consequently, if we apply the Galerkin method but use as weighting functions  $\{u_k^*\}$  rather than  $\{u_k\}$ , then the result corresponds to an adjoint variational principle. The eigenvalue is stationary and we expect a good approximation because first-order errors in approximating  $u$  contribute only second-order errors in approximating the eigenvalue  $\lambda$ .

In the problem at hand the adjoint boundary conditions are the same as the original boundary conditions, so that we can choose  $W_j^* = W_j$ , etc. Then the Galerkin method as applied above is equivalent to application of an adjoint variational principle and the eigenvalue is stationary. This same stationary feature of the Galerkin method is true in other convective instability problems whenever the non-self-adjointness arises from the differential equations rather than the boundary conditions. Thus the eigenvalue is stationary, although the authors do not state this, in the Galerkin calculations done by Krueger & DiPrima (1964), Kurzweg (1964), DiPrima & Pan (1964), Walowit (1966) and Ritchie (1968) for various modifications of the stability of Couette flow between rotating cylinders.

#### 4.3. Numerical results

The results of applying the Galerkin method to (23) are given in table 1 and figures 1 and 2. Results for the third approximation are shown and these generally differed from the second approximation by less than  $\frac{1}{3}\%$ . For the buoyancy problem with no magnetic field ( $M_1 = 0$ ) the successive approximations to Rayleigh number are 1750, 1708.80, 1707.77, compared to an exact value of 1707.762 (Chandrasekhar 1961).

$1 + \chi$	$M_3$	$N_c$	$a_c$	$1 + \chi$	$M_3$	$N_c$	$a_c$
1	1	2570	3.6	10	3	2190	3.5
	2	2270	3.6		1	2960	3.9
	3	2130	3.5		2	2410	3.7
	4	2050	3.4	3	2020	3.5	
	7	1920	3.3	$10^4$	1	3050	4.0
	10	1870	3.3		2	2440	3.7
15	1820	3.2	3		2220	3.6	
3	25	1780	3.2	4	2100	3.5	
	1	2800	3.8	7	1940	3.3	
	2	2360	3.6	10	1870	3.3	
5	3	2180	3.5	15	1820	3.2	
	1	2880	3.9	25	1780	3.2	
	2	2380	3.6				

TABLE 1. Critical stability parameter in the absence of buoyancy forces

The results applicable to the magnetic convection mechanism alone are shown in table 1 and figure 1. The parameters  $M_3$ , representing the departure of the magnetic equation of state from linearity, and  $\chi$ , arising in the magnetic boundary condition cause the critical magnetic number to vary between the limits 1708 and 3050. As the equation of state becomes more non-linear ( $M_3$  large) the fluid layer is destabilized slightly. Eventually the entire problem reduces to the classical Rayleigh problem with  $N_c = 1707.8$ , as  $M_3 \rightarrow \infty$ .

When both magnetic and buoyancy forces can cause convection, the Rayleigh number depends on  $M_1$ , or, equivalently, the critical Rayleigh number and magnetic number,  $N = M_1 R$ , are coupled. For free boundaries we derived the

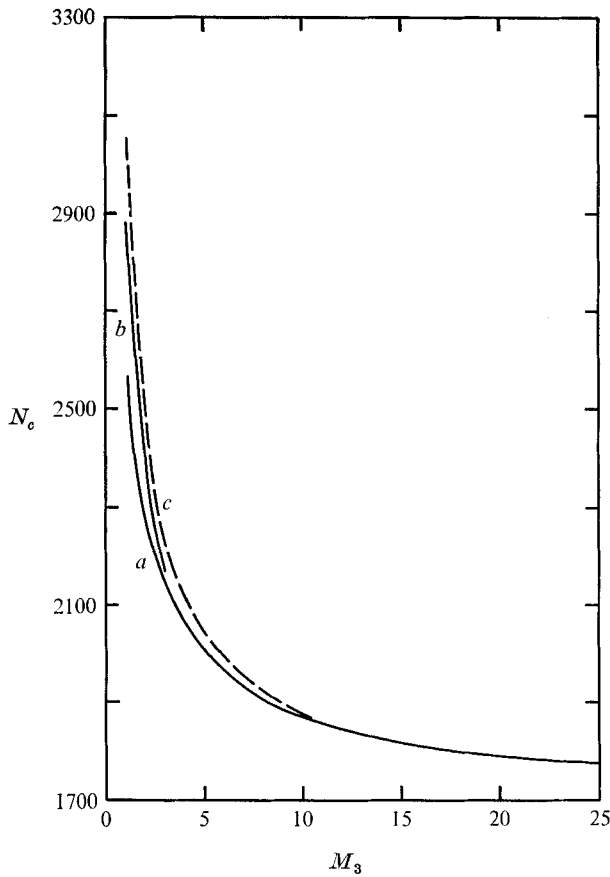


FIGURE 1. Critical stability parameter for magnetic convection mechanism as a function of parameters  $M_3$  and  $\chi$ . Curve: *a*,  $1 + \chi = 1$ ; *b*,  $1 + \chi = 5$ ; *c*,  $\chi \rightarrow \infty$ .

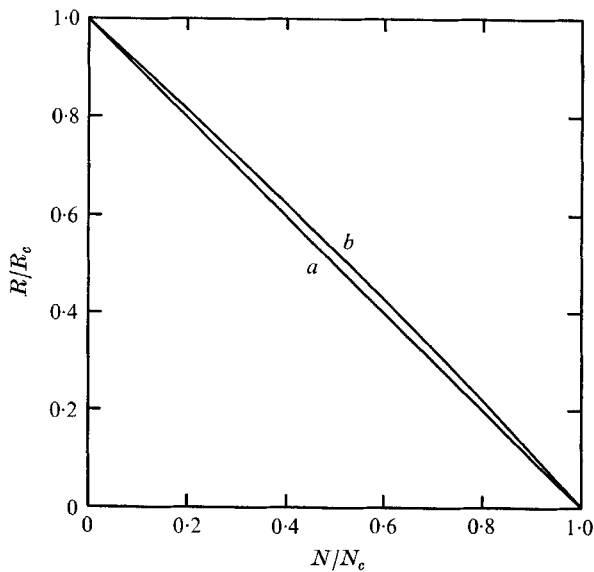


FIGURE 2. Effect of the magnetic mechanism on the Rayleigh number. Curve: *a*,  $M_3 \rightarrow \infty$ ; *b*,  $M_3 = 1$ ,  $1 + \chi = 10^4$ . Values for  $M_3 = 1$ ,  $1 + \chi = 1$ ;  $M_3 = 3$ ,  $1 + \chi = 1$ ;  $M_3 = 1$ ,  $1 + \chi = 5$  are between curves *a* and *b*.

exact form of the coupling in (22). For the present case of rigid boundaries we can no longer derive an exact result, but the first approximation of the Galerkin method gives suggestive results. The equations governing the first approximation can be rearranged into the form

$$\frac{R}{R_c} + \frac{N}{N_c} K f(M_3, \chi, a) = g(a), \quad (26)$$

where

$$f = \frac{\frac{2}{9} + \frac{1}{12} M_3 a^2 + (a/[2(1+\chi)])}{1 + \frac{1}{12} M_3 a^2 + (a/[2(1+\chi)])}, \quad K = K(M_3, \chi).$$

As  $M_3$  approaches infinity,  $f$ ,  $K$  and  $g$  each approach one and we obtain

$$\frac{R}{R_c} + \frac{N}{N_c} = 1. \quad (27)$$

The two convective mechanisms are tightly coupled because they have the same wave-number. For instability, an increase in the forces due to one mechanism makes possible a proportional decrease in the forces due to the other mechanism. It is also clear from the equations that as  $\chi$  becomes large it has less influence on the solution. For higher approximations the exact form of the relationship suggested by (26) must be determined numerically, and the results are shown in figure 2. Even for a wide change in parameter values (27) is followed rather closely.

## 5. Discussion of results and conclusions

We see that the magnetic mechanism alone can induce convection provided the critical stability parameter,  $N_c$ , defined after (18), is above a critical value, which depends on the magnetic equation of state. Consequently, if a ferromagnetic fluid is placed in a fluid layer with a uniform vertical magnetic field, convection will be induced provided the temperature gradient is large enough. We note that the magnetic convection mechanism requires both thermal and magnetic interactions with the surroundings. The mechanism, depending as it does upon the temperature dependence of the magnetic equation of state, is similar to that reported by Poots (1963), Yeung & Yu (1968) and Turnbull (1969) for inducing motion of a dielectric fluid when it is heated and the dielectric coefficient is a function of temperature.

To determine the magnitude of the temperature gradient necessary to experimentally verify the phenomenon, we need to know the magnetic equation of state as a function of external magnetic field and temperature. Dependence on magnetic field is given for typical ferromagnetic fluids by Cowley & Rosensweig (1967). The saturation magnetization as a function of temperature is given by Rosensweig & Kaiser (1967). If we use an external magnetic field of several thousand oersted (1 oersted =  $10^3/4\pi$  A/m), which is sufficient to saturate most ferromagnetic fluids, then  $1 + \chi \simeq 1$ ,  $M_3 \simeq 1$ . The value of  $K$  reported by Rosensweig & Kaiser (1967) is  $K = 0.03$  gauss/ $^{\circ}\text{C}$  (= 30 A/m  $^{\circ}\text{C}$ ) for magnetite in kerosene. Using values of viscosity, thermal conductivity, density, heat capacity and coefficient of expansion appropriate to the pure fluids water, kerosene

and *n*-heptane at 20°C, we obtain the temperature differences shown in table 2 required to cause motion by the buoyancy mechanism and by the combined magnetic and buoyancy mechanism. For such fluids only in thin fluid layers does the magnetic mechanism influence the results. The magnetic forces can be increased by increasing *K*, either by using higher concentrations or by suspending a solid with a higher value of *K*. In very thin layers (less than 1 mm for the fluids described in table 2) only the magnetic forces contribute to convection.

<i>d</i> (cm)	$\beta d$ (°C)					
	Water		Kerosene		<i>n</i> -Heptane	
	Buoyancy alone	Buoyancy and magnetic	Buoyancy alone	Buoyancy and magnetic	Buoyancy alone	Buoyancy and magnetic
1	0.1	0.1	0.05	0.05	0.008	0.008
0.5	0.9	0.9	0.4	0.4	0.06	0.06
0.2	15	7	6	5	1	1
0.1	120	17	51	19	8	5

TABLE 2. Temperature differences required to induce convection

In conclusion, we see that convection can be induced in a ferromagnetic fluid by means of a spatial variation in magnetization which is induced when the magnetization of the fluid depends on temperature and a temperature gradient is established across the layer. Only in thin layers ( $\sim 1$  mm) will the magnetic mechanism predominate over the buoyancy mechanism. This problem represents a new mode of thermo-mechanical interaction arising through the stress tensor, and convective instability theory provides a powerful method of studying such thermo-mechanical interaction.

The author would like to thank Otto C. Faber of NASA, Lewis Research Center for many helpful comments about the properties of ferromagnetic fluids.

## Appendix

The temperature equation for a magnetizable media is derived by combining a thermodynamic treatment with the treatment of the electrodynamics of moving media as presented by Penfield & Haus (1967). We write the total differential for internal energy when it is a function of the independent variables: entropy, specific volume and magnetization:

$$dU = TdS - p'dV + \mu_0 \mathbf{H} \cdot d\mathbf{I}. \quad (\text{A } 1)$$

This is equation (4.120) of Penfield & Haus (1967, p. 85) after changes in notation. The internal energy per unit mass is *U*, entropy per mass is *S*, *V* is specific volume,  $\mathbf{I} = \mathbf{M}V$ , and the pressure includes a magnetic contribution in addition to the usual thermodynamic pressure arising in the absence of magnetic fields.

In an incompressible fluid, of course, the pressure is indeterminate. We assume that (A 1) is valid in a convected co-ordinate system:

$$\frac{dU}{dt} = T \frac{dS}{dt} - p' \frac{dV}{dt} + \mu_0 \mathbf{H} \cdot \frac{d\mathbf{I}}{dt}. \quad (\text{A } 2)$$

The continuity equation gives 
$$\frac{dV}{dt} = \frac{\nabla \cdot \mathbf{v}}{\rho}. \quad (\text{A } 3)$$

Following Penfield & Haus as well as Landau & Lifshitz (1960) we postulate the entropy equation

$$\rho T \frac{dS}{dt} = -\nabla \cdot \mathbf{q} + \Phi, \quad (\text{A } 4)$$

where  $\mathbf{q}$  is the heat flux and  $\Phi$  is the viscous dissipation. Combining (A 2)–(A 4) yields an equation for internal energy.

$$\rho \frac{dU}{dt} = -\nabla \cdot \mathbf{q} + \Phi - p' \nabla \cdot \mathbf{v} + \mu_0 \rho \mathbf{H} \cdot \frac{d\mathbf{I}}{dt}. \quad (\text{A } 5)$$

The remainder of the analysis uses standard techniques of thermodynamics to convert this equation to a temperature equation.

The entropy and magnetic differentials in (A 1) are written in terms of the independent variables  $T$ ,  $V$ ,  $\mathbf{H}$ . The coefficients involving entropy derivatives are evaluated using Maxwell relations obtained from a modified free energy  $A' = U - TS - \mu_0 \mathbf{H} \cdot \mathbf{I}$ . The results, substituted into (A 5), give

$$\left[ \rho C_{VH} - \mu_0 \rho H_i \left( \frac{\partial I_i}{\partial T} \right)_{V, \mathbf{H}} \right] \frac{dT}{dt} + \mu_0 \rho T \left( \frac{\partial I_i}{\partial T} \right)_{V, \mathbf{H}} \frac{dH_i}{dt} = -\nabla \cdot \mathbf{q} + \Phi - T \left( \frac{\partial p'}{\partial T} \right)_{V, \mathbf{H}} \nabla \cdot \mathbf{v}, \quad (\text{A } 6)$$

where

$$\rho \left( \frac{\partial I_i}{\partial T} \right)_{V, \mathbf{H}} = \left( \frac{\partial M_i}{\partial T} \right)_{V, \mathbf{H}}, \quad C_{VH} \equiv \left( \frac{\partial U}{\partial T} \right)_{V, \mathbf{H}} = T \left( \frac{\partial S}{\partial T} \right)_{V, \mathbf{H}} + \mu_0 H_i \left( \frac{\partial I_i}{\partial T} \right)_{V, \mathbf{H}}$$

and (A 6) represents the temperature equation for a ferromagnetic fluid. For an incompressible fluid obeying Fourier's law we get (2). This equation differs slightly from that derived by Neuringer & Rosensweig (1964) and Resler & Rosensweig (1967), but the only difference is the second term in the brackets, which is usually negligible.

The temperature equation can also be derived by beginning with the free energy expression given by Cowley & Rosensweig (1967, equation (A 1)) in place of (A 1) here. With

$$F(\rho, T, B) = F_0(\rho, T) + \int_0^B H(\rho, T, B) dB,$$

the temperature equation is

$$\rho C_{VB} \frac{dT}{dt} - \rho T \left( \frac{\partial H}{\partial T} \right)_{V, B} \frac{dB}{dt} = -\nabla \cdot \mathbf{q} + \Phi - T \left( \frac{\partial p}{\partial T} \right)_{V, B} \nabla \cdot \mathbf{v}, \quad (\text{A } 7)$$

where  $C_{VB} = (\partial U / \partial T)_{V, B}$ . It can be shown that this is equivalent to (A 6).

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