ON THE PROPER BOUNDARY CONDITIONS FOR THE THERMAL ENTRY PROBLEM

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When solving the transport equations for temperature, including convection and diffusion in two dimensions, Heinrich *et al*[1] and Heinrich and Zienkiewicz[2] obtained solutions which oscillated in space. Since these oscillations do not arise in the physical solution, two strategies were suggested for elimination[2,3]. A refined mesh, using smaller elements, can eliminate the oscillations but this solution may be very expensive for large convective terms. Instead an asymmetric weighting function was introduced to cause 'upstream' weighting. Later Gartling[4] showed that the oscillations could also be eliminated by changing the exit boundary condition from one with temperature specified (an essential boundary condition in the terminology of the calculus of variations) to one with no conductive flux (or the natural boundary condition). He gave as the reason for the oscillation the fact that the cell Peclet number was too large to properly resolve steep gradients of temperature. We show in this note that there is a fourth alternative: to increase the length of the calculation domain. The entire problem has arisen because the original, physical problem is for an infinite domain, whereas the mathematical problem is solved on a finite domain. If that domain is too small, steep gradients in temperature occur when the exit temperature is specified. If the domain is large enough, steep gradients do not occur at the exit and either boundary condition suffices: no oscillations appear in the solution.

The thermal entry problem is illustrated in Figure 1. The fluid enters with a dimensionless temperature $T = 0$ and is heated by contact with the solid wall which is maintained at

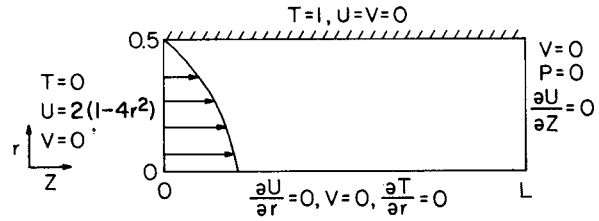

Figure 1. Thermal entry problem in a cylindrical tube

temperature $T = 1$. The goal is to predict the temperature distribution throughout the domain, and in particular to calculate the heat flux to the wall. For convenience here we take the case of fully developed laminar flow when

$$u = 2(1 - 4(r'/D)^2), \qquad v = 0 \tag{1}$$

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where r' is the dimensional radius, D the tube diameter, and u and v are the z and r components of velocity. This problem is called the Graetz problem in the engineering literature. The temperature field is governed by the dimensionlesss transport equation:

$$\mathrm{Pe}\, u\frac{\partial T}{\partial z}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\alpha_z\frac{\partial^2 T}{\partial z^2} \tag{2}$$

$$T(r,0)=0, \qquad T(0{\cdot}5,z)=1, \qquad \partial T/\partial r(0,z)=0 \tag{3}$$

The Peclet number is defined as

$$\mathrm{Pe}=\frac{u'_{\mathrm{avg}}D}{\alpha} \tag{4}$$

where u'_{avg} is the dimensional average velocity, D is the tube diameter, and α is the thermal diffusivity. Both radius, r', and axial distance, z', are made dimensionless by dividing by D. Taking $\alpha_z=0$ neglects axial conduction and leaving $\alpha_z=1$ includes it. With $\alpha_z=1$, equation (2) is elliptic and we need an additional boundary condition at $z=L$:

$$\text{Case I:} \quad T(r,L)=1; \tag{5a}$$

$$\text{Case II:} \quad \partial T/\partial z=0 \tag{5b}$$

There is a discontinuity at the point $z=0$, $r=0{\cdot}5$. This gives rise to very large temperature gradients in the vicinity, and requires a very fine mesh ($\Delta z, \Delta r\to 0$) to resolve. Any errors or oscillations introduced here do not adversely affect the solution downstream. For the purposes of this note we accept these oscillations in the corner and concentrate on the oscillations near the exit plane, $z=L$. Furthermore we plot only the centreline temperatures. The radial average temperature and temperatures at other radii oscillate less but behave in a similar fashion.

It is commonly accepted in the engineering field that axial conduction can be neglected, compared to axial convection, when the Peclet number exceeds 100.[5] In such a case equation (2) reduces to

$$\mathrm{Pe}\, u\frac{\partial T}{\partial z}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{6}$$

The appropriate boundary conditions are (3); equations (5) are not needed, or applicable, when $\alpha_z=0$. Equation (6) clearly demonstrates that the solution is a function of z/Pe, not z or Pe alone; the equation is parabolic. The exact solution to equation (6) is given by Brown[6]:

$$T(r,z)=\sum_{n=1}^{n=\infty} C_n Y_n(r)\exp\left(-\lambda_n^2 z/\mathrm{Pe}\right) \tag{7}$$

The local Nusselt number is often of interest and is defined as

$$\mathrm{Nu}=-\left.\frac{\partial T}{\partial r}\right|_{r=0{\cdot}5}\Big/ T_{\mathrm{avg}} \tag{8}$$

The Nusselt number is infinite at $z=0$, where the temperature is discontinuous at $r=0{\cdot}5$, and approaches an asymptote as $z\to\infty$, reaching the value $\mathrm{Nu}_\infty=3{\cdot}66$ far downstream. The asymptotic formula for large z is shown in Kays:[7]

$$\mathrm{Nu}=\frac{\lambda_0^2}{2}\left(1+\left(1-\frac{\lambda_0^2}{\lambda_1^2}\right)\frac{G_1}{G_0}\exp\left(-(\lambda_1^2-\lambda_0^2)z/Pe\right)\right) \tag{9}$$

We have defined the entry length, L_e, as the distance until the Nusselt number reaches to within 5 per cent of its fully developed value. However, in this work what we are concerned with is the entry length of the fully developed temperature profile, $T(r, L) = 1$, rather than the asymptotic Nusselt number. In the same fashion, the entry length $L_{e,T}$ is defined as the distance for the temperature to reach within 1 per cent of its fully developed profile (e.g. case I). From the paper by Brown[6] these lengths are calculated and compared with L_e at several percent values:

Criterion	L_e	$L_{e,T}$	
10%	0·025 Pe	0·15 Pe	(10)
5%	0·035 Pe	0·20 Pe	(11)
1%	0·05 Pe	0·30 Pe	(12)

In general, the distance it takes the temperature to become fully developed is about six times longer than that for the asymptotic Nusselt number.

To illustrate our point we solve equation (2) with a finite element program. The program solves for the flow field as well and uses 9-node Lagrangian elements for the temperature and two velocities and a 4-node element for pressure. The Galerkin formulation of the equations is described in more detail by Ben-Sabar and Caswell.[8] The meshes used in the calculation are shown in Figure 2. We solve the problem with axial conduction using boundary conditions (5a) or (5b). The case II is a natural boundary condition and is obtained if neither temperature nor external flux is specified on the exit boundary. The same boundary conditions can be used without axial conduction, although they are not then appropriate.

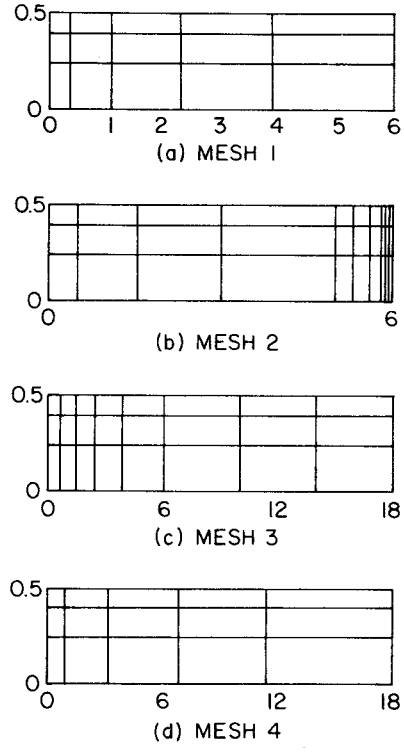

Figure 2. Meshes for study: (a) mesh 1, 3×5, $L = 6$; (b) mesh 2, 3×10, $L = 6$; (c) mesh 3, 3×8, $L = 18$; (d) mesh 4, 3×5, $L = 18$

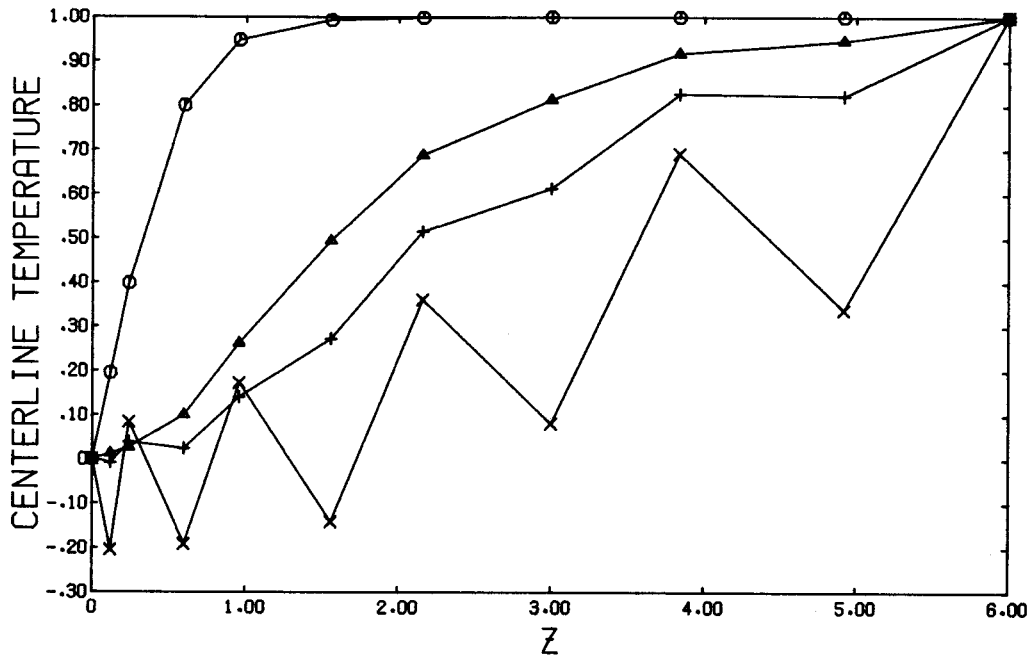

Figure 3. Centreline temperature for short domain, boundary condition (5a). Mesh 1, $\alpha_z = 1$, Pe = 2 ○, 20△, 30 +, 60 ×. Linear interpolation is used to emphasize the numerical oscillation

Figure 3 illustrate the oscillations which occur when the length is taken as 6 diameters long and boundary condition (5a) is used. Note from equation (12) that this distance is less than the thermal entry length, $L_{e,T}$, for Peclet numbers 30, 60 and is greater than the thermal entry length for Pe = 2, 20. Oscillations occur when the calculation domain is less than the entry length. Oscillations do not occur when the calculation domain is greater than the entry length, even when the cell Peclet number is large. The cell Peclet numbers for the last element in the axial direction are given in Table I. Non-oscillatory solutions occur for Pe = 2, 20 even though the cell Peclet number is large.

Table I. Calculation of the cell Peclet number

Figure	Pe†	Pe_{cell}‡
3	2	4·32
	20	43·2
	30	64·8
	60	129·6
4	60	0·36

† $Pe = \dfrac{\langle u \rangle d}{\alpha}$, conventional Peclet number.

‡ $Pe_{cell} = Pe \,.\, \Delta\bar{z} = \dfrac{\langle u \rangle \Delta z}{\alpha}$, axial cell Peclet number for the last element.

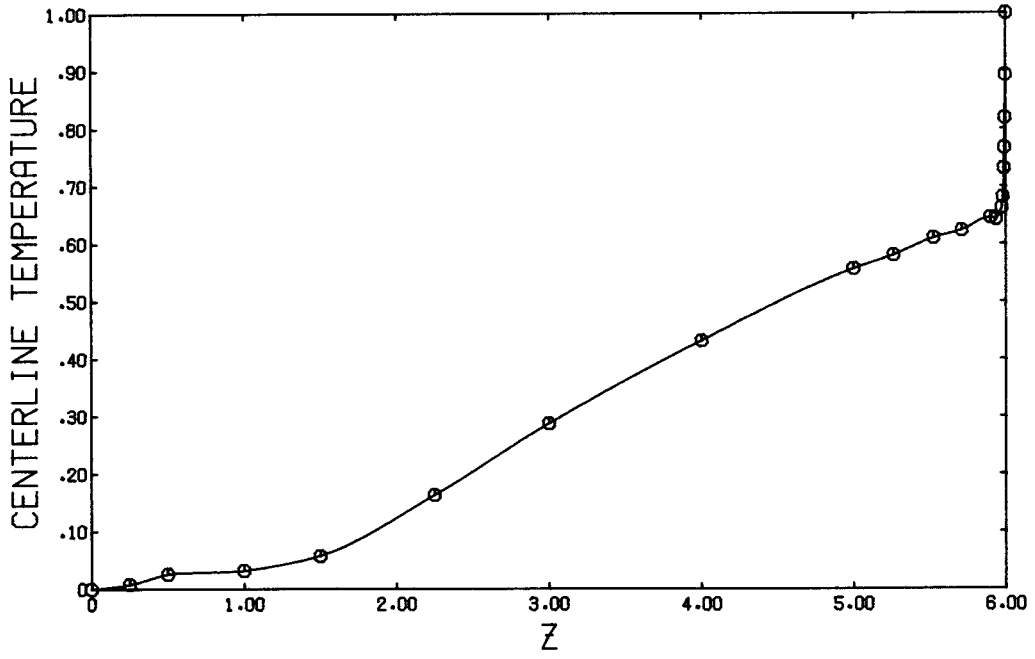

Figure 4. Centreline temperature for short domain, boundary condition (5a) and mesh refinement. Mesh 2, $\alpha_z = 1$, Pe = 60

Now as the Peclet number is increased, the solution to equation (2) for boundary conditions (5a) gives a solution with a steep gradient near $z = L$. The oscillations are apparent for Pe = 30, 60 in Figure 3. A more accurate solution for this case is shown in Figure 4. A fine mesh is used near $z = L$ and the boundary layer nature of the solution is revealed. With such a steep profile the cell Peclet number is important. Roache[9] has shown for the finite difference method that we need

$$\mathrm{Pe}_{\mathrm{cell}} = \mathrm{Pe}\,\Delta z < 2 \tag{13}$$

to prevent oscillations, and Christie *et al*[3] have shown for Galerkin quadratic functions (used here) that we need

$$\mathrm{Pe}_{\mathrm{cell}} = \mathrm{Pe}\,\Delta z < 4 \tag{14}$$

to prevent oscillations.† At the exit $\Delta z = 0{\cdot}006$ in Figure 4, for a cell Peclet number of 0·36, whereas in Figure 3 we had $\Delta z = 2{\cdot}16$, $\mathrm{Pe}_{\mathrm{cell}} = 129{\cdot}6$, and oscillations occurred.

Next change to boundary condition (5b). Solutions for several Peclet numbers are shown in Figure 5. Clearly the oscillations have disappeared. Calculations by Ben-Sabar and Caswell[8] for $\mathrm{Pe} = 10^4$ with these boundary conditions do not have oscillations either. Clearly Gartling was right in showing that the oscillations can be eliminated by changing from boundary condition (5a) to (5b). The cell Peclet number for the calculation with $\mathrm{Pe} = 10^4$ is 2×10^4; oscillations do not appear because there are no steep profiles.

We next repeat the calculations for Pe = 60 on a longer calculation domain, one that is longer than the entry length. Solutions shown in Figure 6 do not oscillate regardless of the boundary

† Jensen and Finlayson[10] have shown that this criterion applies to the solution at the nodes, not necessarily to the interpolated solution in the element.

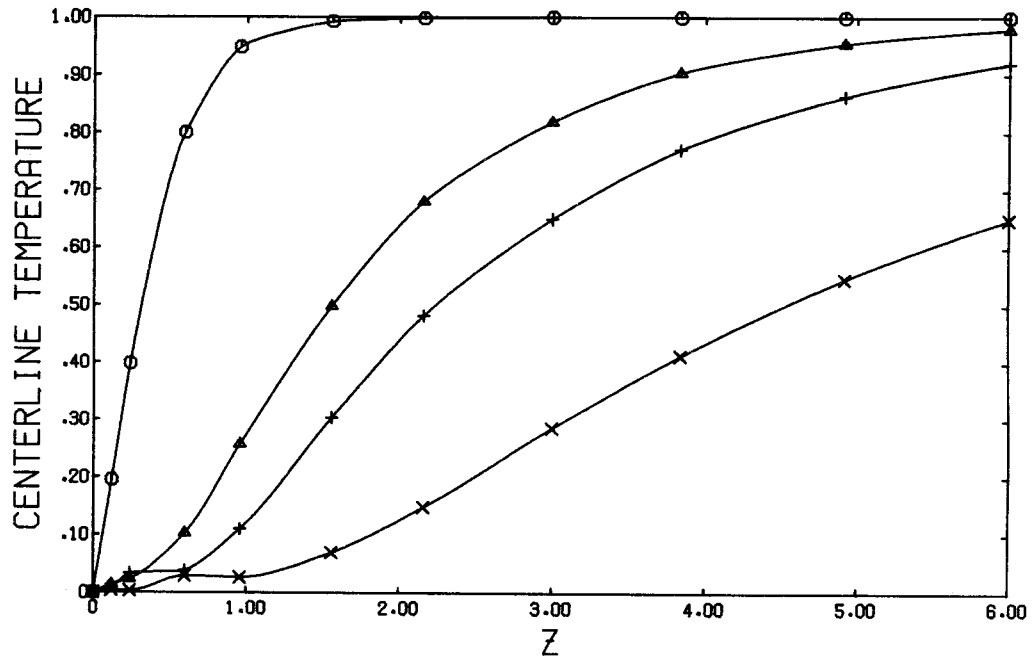

Figure 5. Centreline temperature for short domain, boundary condition (5b). Mesh 1, $\alpha_z = 1$, Pe = 2 ○, 20△, 30 +, 60 ×

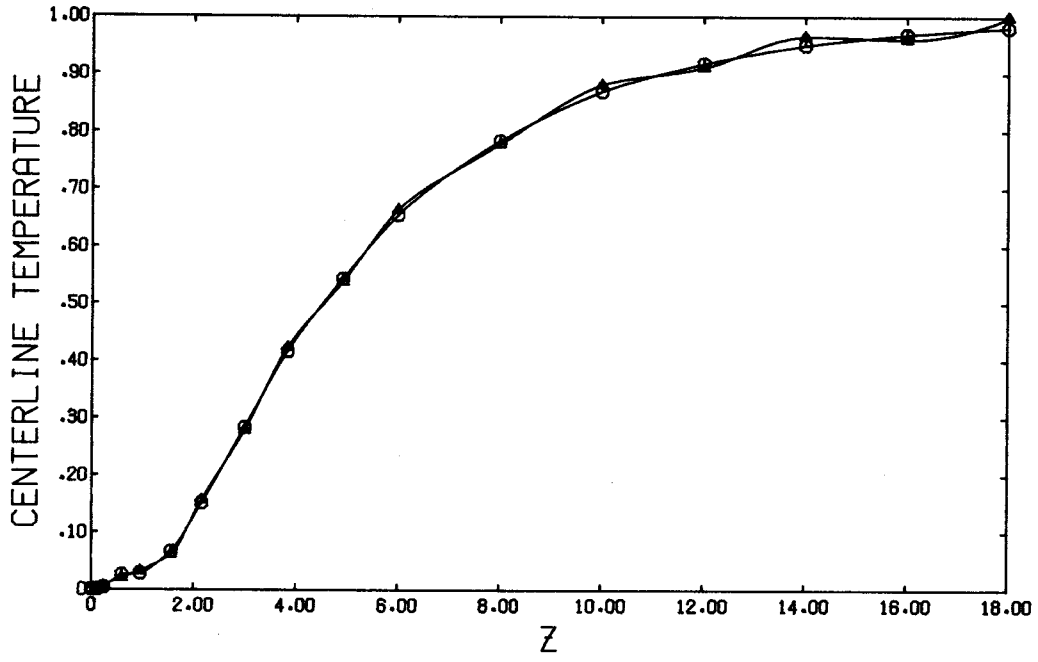

Figure 6. Centreline temperature for long domain. Mesh 3, $\alpha_z = 1$, Pe = 60. Boundary condition (5a) △, boundary condition (5b) . The small fluctuations for boundary condition (5a) are due to the 1 per cent criterion

condition. Again the cell Peclet number is irrelevant because the profiles are not steep. Figure 7 shows the same problem with a coarse mesh at the exit; again no oscillations occur. This means we have a fourth method for eliminating oscillations in this problem: use a calculation grid longer than the entry length.

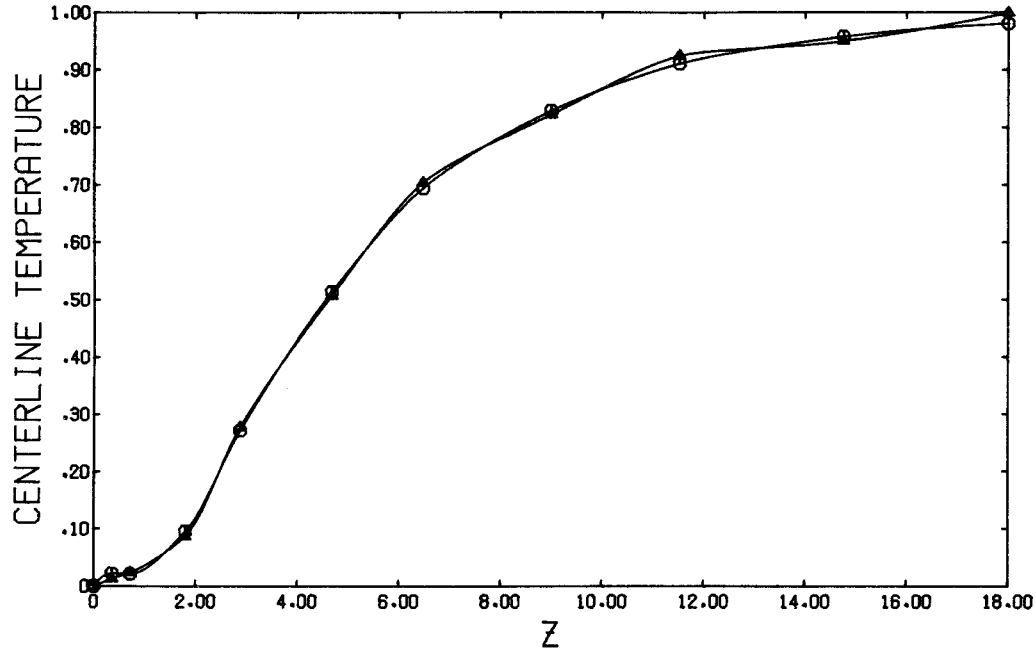

Figure 7. Centreline temperature for long domain. Mesh 4, $\alpha_z = 1$, Pe = 60. Boundary condition (5a) △, boundary condition (5b) ○. The small fluctuations for boundary condition (5a) are due to the 1 per cent criterion

In summary, oscillations are introduced into this problem in the following way. If the calculation grid is long enough, $T = 1$ at the exit and using either boundary condition (5) gives the same results. If the calculation grid is shorter than the entry length and boundary condition (5b) is used, a solution is obtained which does not oscillate. It also does not have steep gradients. Furthermore the temperature is not one at the exit. Suppose it is 0·8. If we change to boundary condition (5a) and require $T = 1$ at the exit, the problem responds by forcing a rapid change from the value 0·8 just inside the exit boundary to 1·0 at the exit boundary (see Figure 4). A temperature boundary layer is thus developed. A steep profile now exists and the cell Peclet number must be kept small to avoid oscillations, or else an asymmetric weighting function must be introduced. All these problems, however, are because the boundary condition is incorrect. We cannot specify the temperature of an outflow region because physically we have no way to maintain it. If the calculation domain is long enough, the specification of $T = 1$ as an exit condition is acceptable since the solution has $T = 1$ there anyway. If the calculation domain is too short, the specification of any temperature (except the exact value) causes oscillations, as shown in Figure 8.

We conclude that the more reliable solution to this problem is for a calculation domain longer than the entry length and a flux boundary condition (5b). The temperature can be specified on an

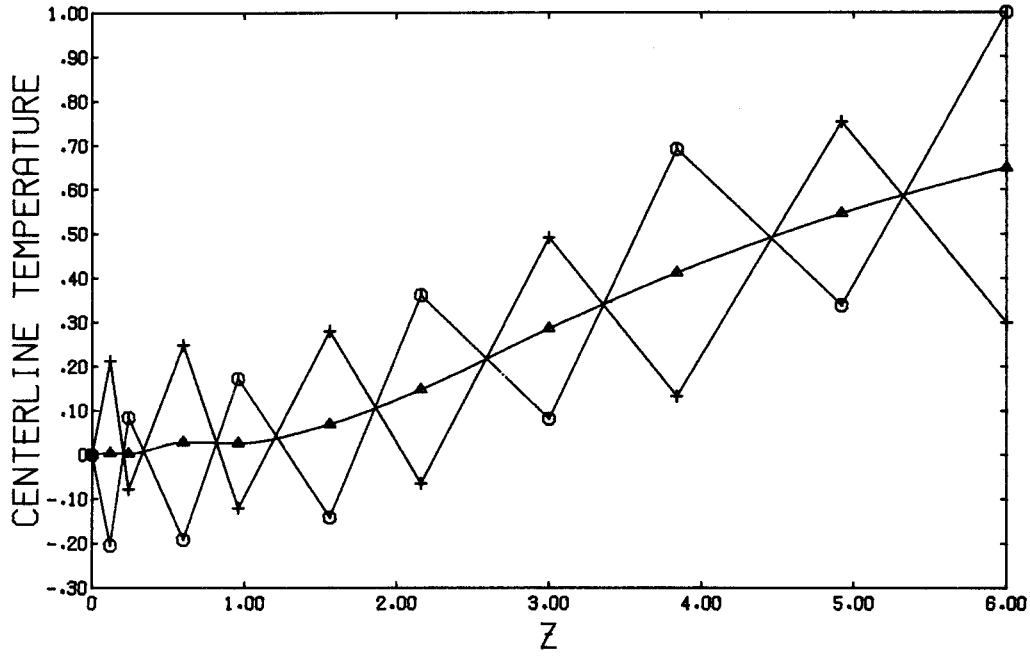

Figure 8. Centreline temperature for short domain and improper boundary value. Mesh 1, $\alpha_z = 1$, Pe = 60. (1) Boundary condition (5a) ○, (2) boundary condition (5b) △, (3) boundary condition $T(r, L) = 2T(r, L)_{(2)} - 1$, +. Linear interpolation is used to emphasize the numerical oscillations

outflow boundary (5a) only if careful analysis ensures the temperature is the correct one, usually because the domain is long enough that the proper temperature profile is reached.

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