

Simulated Method of Moments Estimation of Stochastic Volatility Models

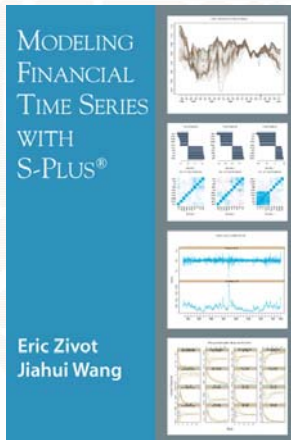
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Distinguished Scholar

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- “Next Generation Component Software for Simulation-Based Econometric Estimation”
 - Maximum Simulated Likelihood Estimation
 - Method of Simulated Moments
- Applications
 - Discrete choice models
 - **Continuous time financial models**
 - Nonlinear time series models

Modeling Financial Time Series with S-PLUS



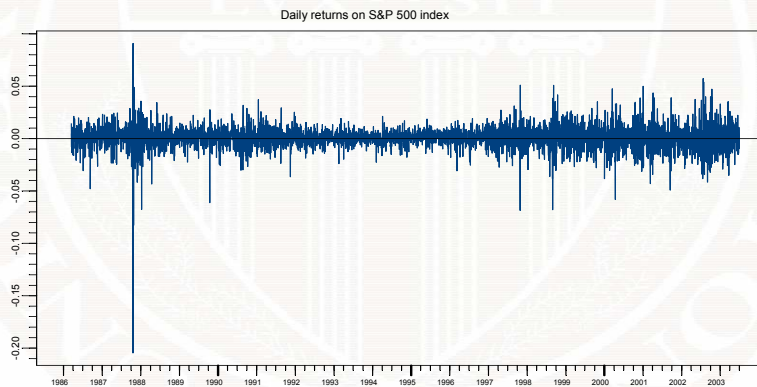
➤ New Chapters in *Second Edition*

- Copulas
- Nonlinear Models
- Continuous-Time Models
- Generalized Method of Moments
- Semi-nonparametric Conditional Density Models
- Efficient Method of Moments

Agenda

- Discrete-time stochastic volatility models
- Continuous-time stochastic volatility models
- Simulating from continuous time models
- Simulated method of moments estimation
- Estimation of continuous time models using S-PLUS

Daily Returns on S&P 500



Discrete Time Stochastic Volatility Models

$$y_t = \sigma_t z_t$$

$$\ln \sigma_t^2 = \omega + \beta \ln \sigma_{t-1}^2 + \sigma_u u_t$$

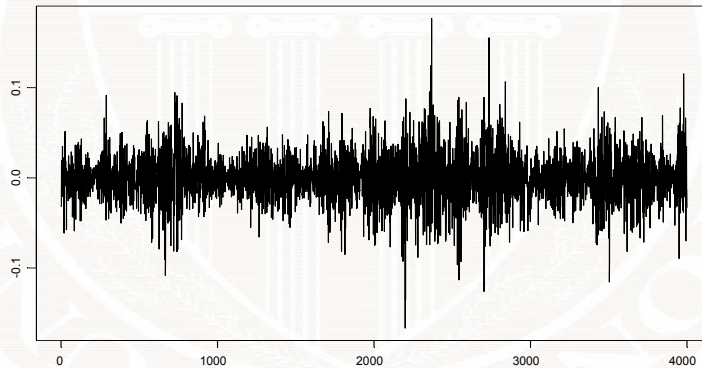
$$(z_t, u_t)' \sim iid N(0, I_2)$$

$$-1 < \beta < 1, \sigma_u > 0$$

Andersen, Chung and Sorensen (1999): "Efficient Method of Moments Estimation of a Stochastic Volatility Model: A Monte Carlo Study," *Journal of Econometrics*.

Simulated SV Data

$$\alpha = -0.147, \beta = 0.98, \sigma_u = 0.166$$



Continuous Time Models in Finance

U_t = state variables

$$dU_t = A(U_t, \rho)dt + B(U_t, \rho)dW_t$$

$A(U_t, \rho)$ = drift

$B(U_t, \rho)$ = diffusion

ρ = model parameters

W_t = Wiener process

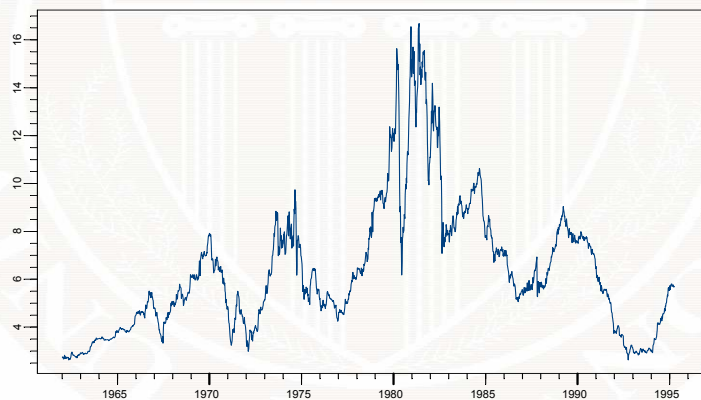
Continuous Time Models in Finance

The state variables are determined by solving the SDE

$$U_t - U_0 = \int_0^t A(U_s, \rho) ds + \int_0^t B(U_s, \rho) dW_s$$

Observables y_t are regarded as discretely sampled observations from part of the above system. Lo (1988 *ET*) showed that the likelihood of the observed process y_t given ρ is generally not available in closed form.

3 Month U.S. T-Bill Rates



Stochastic Volatility Models

One factor generalized CIR model for short rate r_t

$$dr_t = \kappa(\mu - r_t)dt + \sigma r_t^\gamma dW_t$$

$$U_t = r_t, \rho = (\kappa, \mu, \sigma, \gamma)'$$

$$A(U_t, \rho) = \kappa(\mu - r_t), B(U_t, \rho) = \sigma r_t^\gamma$$

Andersen, T. and J. Lund (1997). "Estimating Continuous-Time Stochastic Volatility Models of the Short-term Interest Rate," *Journal of Econometrics*.

Stochastic Volatility Models

Two factor generalized CIR model for short rate r_t

$$dr_t = \kappa_1(\mu - r_t)dt + \exp\left(\frac{v_t}{2}\right)r_t^\gamma dW_{1t},$$

$$dv_t = \kappa_2(\alpha - v_t)dt + \xi dW_{2t},$$

$$A(U_t, \rho) = \begin{pmatrix} \kappa_1(\mu - r_t) \\ \kappa_2(\alpha - v_t) \end{pmatrix}, B(U_t, \rho) = \begin{pmatrix} \exp\left(\frac{v_t}{2}\right)r_t^\gamma & 0 \\ 0 & \xi \end{pmatrix}$$

$$U_t = (r_t, v_t)', W_t = (W_{1t}, W_{2t})', \rho = (\kappa_1, \mu, \gamma, \kappa_2, \alpha, \xi)'$$

Affine Term Structure Model

Dai and Singleton (2000), *Journal of Finance*

$$r_t = \delta_0 + \delta_F' F_t$$

$$dF_t = K(\theta - F_t)dt + \Sigma\sqrt{S_t}dW_t$$

$$S_{ii,t} = \alpha_i + \beta_i' F_t$$

$$\Lambda_t = \sqrt{S_t}\lambda = \text{market price of risk}$$

Simulation Methods for SDEs

Euler's method: Iterate for small delta (Δ)

$$U_{\Delta} - U_0 = A(U_0)\Delta + B(U_0)(W_{\Delta} - W_0)$$

Sum over increments

$$\sum_{i=1}^{t/\Delta} [U_{i\Delta} - U_{(i-1)\Delta}] = \sum_{i=1}^{t/\Delta} A[U_{(i-1)\Delta}]\Delta + \sum_{i=1}^{t/\Delta} B[U_{(i-1)\Delta}]\Delta[W_{i\Delta} - W_{(i-1)\Delta}]$$

Limit as $\Delta \rightarrow 0$

$$U_t - U_0 = \int_0^t A(U_s)ds + \int_0^t B(U_s)dW_s$$

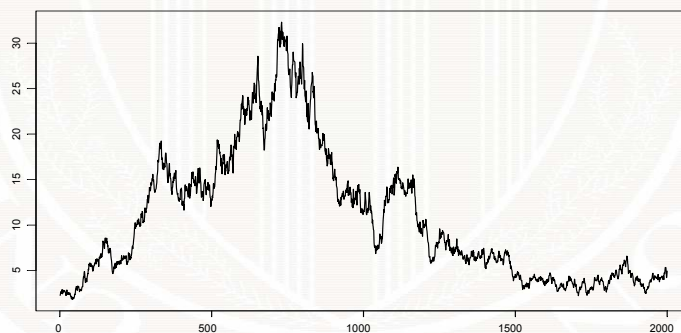
Simulate GCIR Process in S-PLUS

```
# calibrate to estimates obtained by Andersen and Lund
> rho.gcir = c(0.082,6.279,0.670,0.676)
> rho.gcir.names = c("k","mu","sigma","gamma")
> ird.gcir.aux = euler.pcode.aux(drift.expr=expression(k*(mu-X)),
+                               diffuse.expr=expression(sigma*X^gamma),
+                               rho.names = rho.gcir.names,
+                               t.per.sim = 1/52, ndt = 25,
+                               seed = 123, X0 = 6,
+                               lbound=0,ubound=100)
> gcir.sim = euler1d.pcode.gensim(rho.gcir,n.sim=2000,
+                               n.var=1,n.burn=1000,aux=gcir.aux)
```

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Simulated GCIR Process

$$\kappa = 0.082, \mu = 6.279, \sigma = 0.670, \gamma = 0.676$$

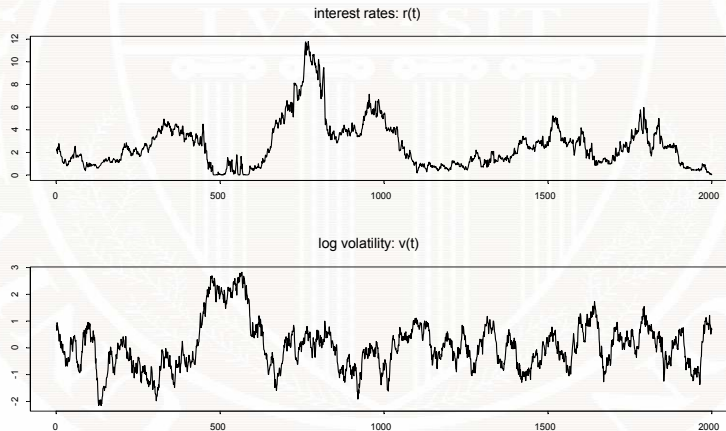


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Simulate 2 Factor Diffusion

```
# calibrate to estimates obtained by Andersen and Lund
> rho.tf = c(0.163,5.95,0.544,1.04,5-0.282,1.27)
> rho.tf.names = c("k1","mu","gamma","k2","alpha","xi")
> tf.aux=euler.pcode.aux(rho.names=rho.tf.names,
+ drift.expr=expression(k1*(mu - X[1]),
+                        k2*(alpha-X[2])),
+ diffuse.expr=expression((exp((X[2]-5)*0.5))*X[1]^gamma,
+                          0.0, 0.0, xi),
+ N=2, M=2, t.per.sim=1/52, ndt = 25, seed = 123,
+ X0 = c(6,5-0.28), lbound=0, ubound=100, returnc=c(T,T))
> tf.sim = euler.pcode.gensim(rho=rho.tf, n.sim=2000, n.var=2,
+                             n.burn=100, aux=tf.aux)
```

Simulated 2 Factor Process



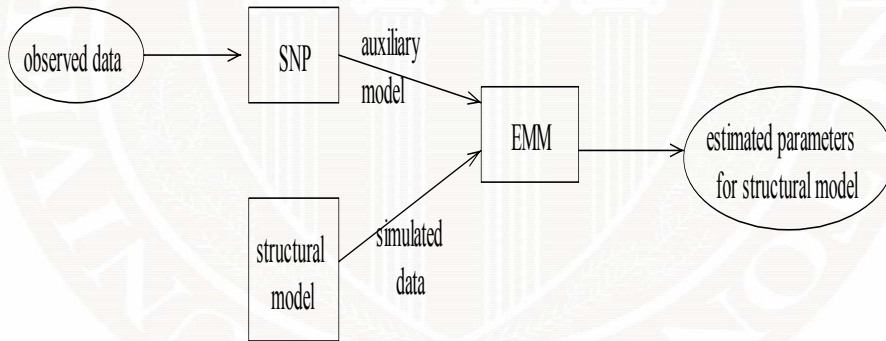
Estimation of Dynamical Systems with Unobserved States

- Standard statistical methods, both classical and Bayesian, are usually not applicable either because it is not practicable to obtain the density of the state vector or because the integration required to eliminate unobserved states from the likelihood is infeasible.
- On a case-by-case basis, statistical methods are sometimes available. However, the purpose here is to describe methods that are generally applicable.
- Simulating the evolution of the state vector is often practicable. The methods described here rely on this.

Efficient Method of Moments

- Gallant, A. R., and G. Tauchen (1996). "Which Moments to Match?," *Econometric Theory*
- Gallant, A. R., and J.R. Long (1997). "Estimating Stochastic Differential Equations Efficiently by Minimum Chi-Squared," *Boimetrika*.
- Gallant, A.R., and G. Tauchen (2004). "Simulated Score Methods and Indirect Inference for Continuous-time Models," forthcoming in the *Handbook of Financial Econometrics*.
- <http://www.econ.duke.edu/~get/emm.htm>

EMM Flow Diagram



Simulated Method of Moments

- For ρ in the parameter space, the model generating the data is presumed to be stationary, ergodic and convenient to simulate
- For any lag length L and parameter setting ρ there exists a stationary density for observables $p(y, \rho)$. An unconditional expectation of a function Ψ can be computed by generating a long simulation $\{y_t(\rho)\}$ and averaging

$$E_{\rho}[\psi(y_{-L}, \dots, y_0)] = \frac{1}{N} \sum_{t=0}^N \psi(y_{t-L}(\rho), \dots, y_t(\rho))$$

Simulated Method of Moments

For a QMLE of an auxiliary model with conditional density $f(y_t|x_{t-1}, \theta)$

$$\tilde{\theta}_n = \arg \max_{\theta} \frac{1}{n} \sum_{t=1}^n \log f(y_t | x_{t-1}, \theta)$$

a sample average satisfies

$$\frac{1}{n} \sum_{t=1}^n \frac{\partial}{\partial \theta} \log f(y_t | x_{t-1}, \tilde{\theta}_n) = 0$$

Simulated Method of Moments

Therefore, a large simulation from the model with parameters ρ should satisfy the estimating equations

$$m(\rho, \tilde{\theta}_n) = \frac{1}{N} \sum_{t=1}^N \frac{\partial}{\partial \theta} \log f(y_t(\rho) | x_{t-1}(\rho), \tilde{\theta}_n) \approx 0$$

If the dimension of θ is greater than the dimension of ρ then estimate ρ using GMM

$$\hat{\rho}_n = \arg \min_{\rho} m(\rho, \tilde{\theta}_n)' (\tilde{I}_n)^{-1} m(\rho, \tilde{\theta}_n)$$

EMM Diagnostics

Under the correct model specification

$$J(\hat{\rho}_n, \tilde{\theta}_n) = n \cdot m(\hat{\rho}_n, \tilde{\theta}_n)' (\tilde{I}_n)^{-1} m(\hat{\rho}_n, \tilde{\theta}_n) \stackrel{A}{\sim} \chi(q)$$

$$q = \dim(\theta) - \dim(\rho)$$

If J-stat is large, t-ratios help diagnose misspecification

$$t(\hat{\rho}_n, \tilde{\theta}_n) = \hat{S}_n^{-1} \sqrt{n} \cdot m(\hat{\rho}_n, \tilde{\theta}_n) \stackrel{A}{\sim} N(0, I)$$

Advantages of EMM over GMM

- EMM automatically generates moments from SNP auxiliary model
 - GMM requires user-specified moments
- EMM uses moments that mimic infeasible MLE
 - GMM uses arbitrary moments
- EMM provides model diagnostics for each moment
 - GMM only provides model diagnostic based on weighted average of all moments

SNP Model

- SNP is a **Semi-NonParametric** method, based on an expansion in Hermite functions, for estimation of the one-step-ahead conditional density $f(y_t|x_{t-1}, \theta)$.
- It consists of applying classical parametric estimation and inference procedures to models derived from nonparametric truncated series expansions.
- Estimation of SNP models entails using a standard maximum likelihood procedure together with a model selection strategy that determines the appropriate degree of the nonparametric expansion. Under reasonable regularity conditions, the estimator is consistent and efficient.

Overview of SNP Methodology

Location-scale transformation for a multivariate innovation z

$$y_t = R_{x_{t-1}} z + \mu_{x_{t-1}}$$

Innovation density (Hermite expansion)

$$h_K(z) = \frac{[P(z, x)]^2 \phi(z)}{\int [P(u, x)]^2 \phi(u) du}$$

$P(z, x)$ = polynomial in z of degree K whose coefficients are polynomials of degree K_x in x

Overview of SNP Methodology

Location function (VAR)

$$\mu_{x_{t-1}} = b_0 + Bx_{t-1}, \quad x_{t-1} = (y_{t-L}, \dots, y_{t-1})$$

ARCH scale function

$$\text{vech}(R_{x_{t-1}}) = r_0 + \sum_{i=1}^{L_r} P_{(i)} |y_{t-1-L_r+i} - \mu_{x_{t-2-L_r+i}}|$$

GARCH scale function

$$\text{vech}(R_{x_{t-1}}) = r_0 + \sum_{i=1}^{L_r} P_{(i)} |y_{t-1-L_r+i} - \mu_{x_{t-2-L_r+i}}| + \sum_{i=1}^{L_g} \text{diag}(G_{(i)}) R_{x_{t-2-L_g+i}}$$

Fit SNP Model to Simulated Data from Discrete Time SV Model

$$y_t = \sigma_t z_t$$

$$\ln \sigma_t^2 = \omega + \beta \ln \sigma_{t-1}^2 + \sigma_u u_t$$

$$(z_t, u_t)' \sim iid N(0, I_2)$$

$$-1 < \beta < 1, \quad \sigma_u > 0$$

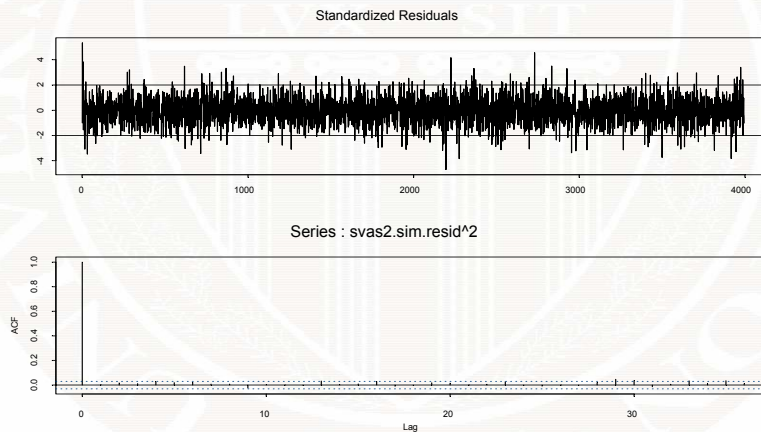
$$\alpha = -0.147, \beta = 0.98, \sigma_u = 0.166$$

Fit SNP Model to Simulated Discrete-Time SV Model

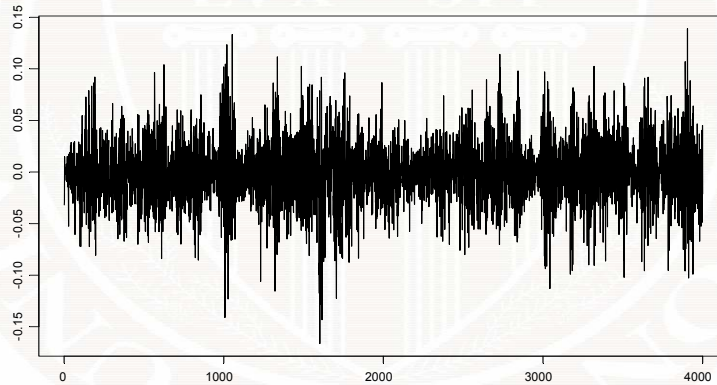
Model: Gaussian GARCH Conditional Variance Coefficients:

		s0	arch(1)	garch(1)
Conditional Mean	coef	0.0332	0.1514	0.8542
Coefficients:	(std.err)	0.0024	0.0080	0.0058
	(t.stat)	14.0586	18.8740	148.3114
mu	coef	0.0191		
	(std.err)	0.0105		
	(t.stat)	1.8273		

SNP Diagnostics: Residuals



SNP Diagnostics: Simulation from Fitted SNP Model



EMM Fit for Simulated Data

Coefficients: $\alpha = -0.147, \beta = 0.98, \sigma_u = 0.166$

	Value	Std. Error	95% Conf. Int.	
alpha	-0.2631	0.13018	-0.3839	-0.1330
beta.t	3.3228	0.50730	2.8523	3.8301
sigma.u	0.1950	0.04636	0.1486	0.2378

Note: $\beta = \exp(\text{beta.t}) / (1 + \exp(\text{beta.t})) = 0.9652$

Final optimization:

Convergence: relative function convergence

Iterations: 30

Score at final iteration: 0.1688

EMM Diagnostics

EMM objective at final iteration: 0.1688

P-Value: 0.6812 on 1 degrees of freedom

Score Information:

	Mean Score	Std. Error	t-ratio	Adj. Std. Error	t-ratio
mu	0.8230	2.004	0.410711	2.0029	0.4109
s0	2.4909	16.076	0.154946	6.0634	0.4108
arch(1)	0.3551	8.196	0.043326	0.8647	0.4107
garch(1)	0.1139	11.928	0.009551	0.2777	0.4102

SNP Model for S&P 500 Data

Model: Semiparametric GARCH

Hermite Polynomial Coefficients:

z^0	z^1	z^2	z^3	z^4	z^5	z^6	z^7	z^8
1	-0.0938	-0.4103	0.0356	0.0925	-0.0049	-0.0082	0.0001	0.0002

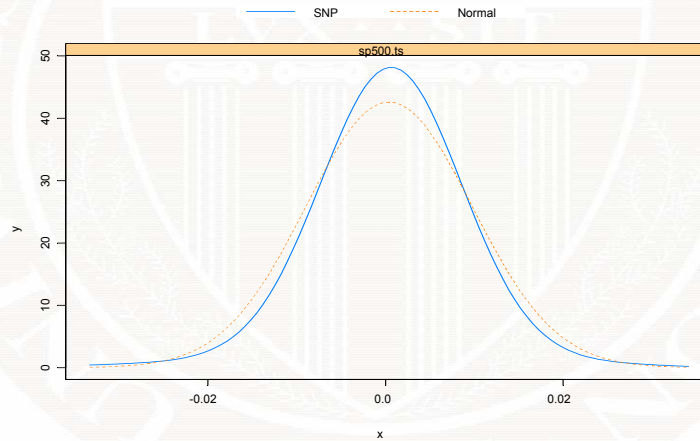
Conditional Mean Coefficients:

mu	ar(1)
0.1032	0.0202

Conditional Variance Coefficients:

s0	arch(1)	garch(1)
0.0122	0.162	0.9105

SNP Conditional Density



EMM Fit of SV Model for S&P 500

Coefficients:

	Value	Std. Error	95% Conf. Int.
alpha	-0.2945	0.13010	-0.4059 -0.1644
beta.t	3.4229	0.45495	3.0318 3.8778
sigma.u	0.1759	0.03137	0.1445 0.1921

Note: $\beta = \exp(\text{beta.t}) / (1 + \exp(\text{beta.t})) = 0.9684$

Final optimization:

Convergence: relative function convergence

Iterations: 24

EMM objective at final iteration: 71.88

P-Value: 1.916e-011 on 10 degrees of freedom

EMM Diagnostics

Score Information:

	Mean Score	Std. Error	t-ratio	Adj. Std. Error	t-ratio
z^1	4.1963	2.4735	1.6965	2.2920	1.8308
z^2	4.5718	3.7849	1.2079	2.5574	1.7877
z^3	-13.2048	8.6945	-1.5188	8.4730	-1.5585
z^4	42.3197	21.7408	1.9466	16.7711	2.5234
z^5	-196.1537	65.0210	-3.0168	63.9647	-3.0666
z^6	385.8924	213.5628	1.8069	196.3508	1.9653
z^7	-2367.8394	760.7266	-3.1126	755.8569	-3.1327
z^8	4874.8652	2856.7967	1.7064	2783.6964	1.7512
mu	4.2844	1.4805	2.8938	0.9612	4.4571
ar(1)	0.2232	0.9803	0.2277	0.9763	0.2286
s0	-16.1841	14.6437	-1.1052	12.1671	-1.3301
arch(1)	0.8420	8.0812	0.1042	6.5281	0.1290
garch(1)	4.4057	15.6225	0.2820	12.5225	0.3518

Fit SNP Model to simulated Data from GCIR

$$dr_t = \kappa(\mu - r_t)dt + \sigma r_t^\gamma dW_t$$

$$\kappa = 0.082, \mu = 6.279, \sigma = 0.670, \gamma = 0.676$$



Fit SNP Model to Simulated GCIR Data

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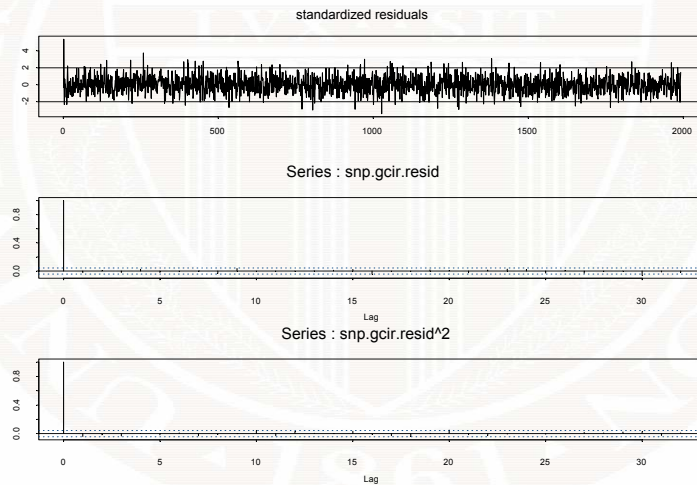
Model: Nonlinear Nonparametric
Information Criteria:
      BIC      HQ      AIC      logL
Hermite Polynomial Coefficients:  -1.4452 -1.4523 -1.4564 2909.204
      x^0      x^1
z^0  1.0000  0.2044
z^1 -0.3793 -0.3122

Convergence Type:
relative function convergence
number of iterations: 22

Conditional Mean Coefficients:
      mu  ar(1)
-0.0487 0.9537

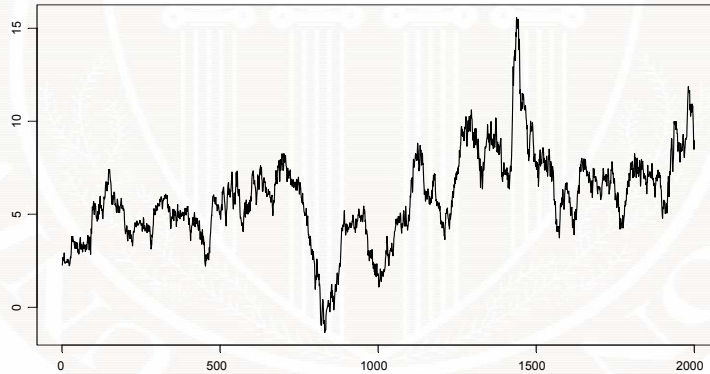
Conditional Variance Coefficients:
      s0 arch(1) garch(1)
-0.0017 -0.0814  0.9058
    
```

SNP Diagnostics: Residuals



SNP Diagnostics: Simulation

simulation from SNP 1111010 model



EMM Estimation of Simulated Data from GCIR Model

Coefficients: $\kappa = 0.082$, $\mu = 6.279$, $\sigma = 0.670$, $\gamma = 0.676$

	Value	Std. Error	95% Conf. Int.	
k	0.3112	0.1345	0.1781	0.4903
mu	6.1965	0.7397	5.4086	6.8541
sigma	0.9628	0.1842	0.7811	1.1130
gamma	0.4628	0.1137	0.3741	0.5262

Final optimization:

Convergence: both x- and relative function convergence

Iterations: 38

EMM objective at final iteration: 4.334

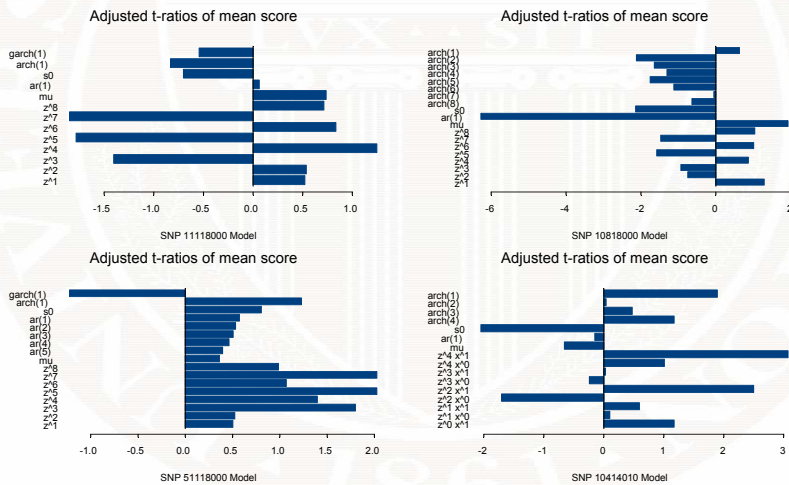
P-Value: 0.3627 on 4 degrees of freedom

EMM Diagnostics

Score Information:

	Mean Score	Std. Error	t-ratio	Adj. Std. Error	t-ratio
$z^0 x^1$	-0.3429	0.7511	-0.4565	0.6252	-0.5485
$z^1 x^0$	0.1815	2.0915	0.0868	1.2303	0.1476
$z^1 x^1$	-0.3733	1.7436	-0.2141	1.0502	-0.3554
μ	3.9899	29.1914	0.1367	21.3918	0.1865
$ar(1)$	-9.6053	28.9383	-0.3319	23.1921	-0.4142
$s0$	-304.3415	371.0627	-0.8202	227.0778	-1.3403
$arch(1)$	-21.3935	15.2993	-1.3983	13.1178	-1.6309
$garch(1)$	24.5928	18.5855	1.3232	14.4695	1.6996

EMM Estimation of GCIR Model for T-Bill Rates: Diagnostic Moments



Conclusions

- EMM is a powerful, flexible and efficient simulation-based estimation method for a wide variety of discrete-time and continuous-time stochastic volatility models
- New user-friendly S-PLUS implementation of Gallant and Tauchen's FORTRAN code exposes the power of EMM for general use