Factor Model Risk Analysis

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- Risk measures
- Factor Risk Budgeting
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Factor Model Specification

Factor models for asset returns have the general form

$$\begin{aligned} R_{it} &= \alpha_i + \beta_{1i} f_{1t} + \beta_{2i} f_{2t} + \cdots + \beta_{Ki} f_{Kt} + \varepsilon_{it} \\ &= \alpha_i + \boldsymbol{\beta}_i' \mathbf{f}_t + \varepsilon_{it} \end{aligned} \tag{1}$$

- R_{it} is the simple return (real or in excess of the risk-free rate) on asset i $(i = 1, \ldots, N)$ in time period t $(t = 1, \ldots, T)$,

- f_{kt} is the k^{th} *common factor* $(k = 1, \ldots, K)$,

- β_{ki} is the *factor loading* or *factor beta* for asset i on the k^{th} factor,

- ε_{it} is the asset *specific factor*.

Assumptions

1. The factor realizations, $\mathbf{f}_t$, are stationary with unconditional moments

$$\begin{aligned} E[\mathbf{f}_t] &= \boldsymbol{\mu}_f \\ cov(\mathbf{f}_t) &= E[(\mathbf{f}_t - \boldsymbol{\mu}_f)(\mathbf{f}_t - \boldsymbol{\mu}_f)'] = \boldsymbol{\Omega}_f \end{aligned}$$

2. Asset specific error terms, ε_{it}, are uncorrelated with each of the common factors, f_{kt},

$$cov(f_{kt}, \varepsilon_{it}) = 0, \text{ for all } k, \ i \text{ and } t.$$

3. Error terms ε_{it} are serially uncorrelated and contemporaneously uncorrelated across assets

$$\begin{aligned} cov(\varepsilon_{it}, \varepsilon_{js}) &= \sigma_i^2 \text{ for all } i = j \text{ and } t = s \\ &= 0, \text{ otherwise} \end{aligned}$$

Cross Section Regression

The multifactor model (1) may be rewritten as a *cross-sectional* regression model at time t by stacking the equations for each asset to give

$$\begin{aligned}
\underset{(N\times 1)}{\mathbf{R}_t} &= \underset{(N\times 1)}{\boldsymbol{\alpha}} + \underset{(N\times K)}{\mathbf{B}}\underset{(K\times 1)}{\mathbf{f}_t} + \underset{(N\times 1)}{\boldsymbol{\varepsilon}_t},\ t = 1,\ldots,T \qquad (2) \\
\underset{(N\times K)}{\mathbf{B}} &= \begin{bmatrix} \boldsymbol{\beta}_1' \\ \vdots \\ \boldsymbol{\beta}_N' \end{bmatrix} = \begin{bmatrix} \beta_{11} & \cdots & \beta_{1K} \\ \vdots & \ddots & \vdots \\ \beta_{N1} & \cdots & \beta_{NK} \end{bmatrix} \\
E[\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'|\mathbf{f}_t] &= \mathbf{D} = diag(\sigma_1^2,\ldots,\sigma_N^2)
\end{aligned}$$

Note: Cross-sectional heteroskedasticity

Time Series Regression

The multifactor model (1) may also be rewritten as a *time-series* regression model for asset i by stacking observations for a given asset i to give

$$\underset{(T\times 1)}{\mathbf{R}_i} = \underset{(T\times 1)}{\mathbf{1}_T}\underset{(1\times 1)}{\alpha_i} + \underset{(T\times K)}{\mathbf{F}}\underset{(K\times 1)}{\boldsymbol{\beta}_i} + \underset{(T\times 1)}{\boldsymbol{\varepsilon}_i},\ i = 1,\ldots,N \quad (3)$$

$$\underset{(T\times K)}{\mathbf{F}} = \begin{bmatrix} \mathbf{f}_1' \\ \vdots \\ \mathbf{f}_T' \end{bmatrix} = \begin{bmatrix} f_{11} & \cdots & f_{Kt} \\ \vdots & \ddots & \vdots \\ f_{1T} & \cdots & f_{KT} \end{bmatrix}$$

$$E[\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i'] = \sigma_i^2\mathbf{I}_T$$

Note: Time series homoskedasticity

Expected Return ($\alpha - \beta$) Decomposition

$$E[R_{it}] = \alpha_i + \boldsymbol{\beta}_i' E[\mathbf{f}_t]$$

- $\boldsymbol{\beta}_i' E[\mathbf{f}_t] =$ explained expected return due to systematic risk factors

- $\alpha_i = E[R_{it}] - \boldsymbol{\beta}_i' E[\mathbf{f}_t] =$ unexplained expected return (abnormal return)

Note: Equilibrium asset pricing models impose the restriction $\alpha_i = 0$ (no abnormal return) for all assets $i = 1, \ldots, N$

Covariance Structure

Using the cross-section regression

$$\underset{(N\times 1)}{\mathbf{R}_t} = \underset{(N\times 1)}{\boldsymbol{\alpha}} + \underset{(N\times K)}{\mathbf{B}}\underset{(K\times 1)}{\mathbf{f}_t} + \underset{(N\times 1)}{\boldsymbol{\varepsilon}_t}, \ t = \mathbf{1}, \ldots, T$$

and the assumptions of the multifactor model, the $(N \times N)$ covariance matrix of asset returns has the form

$$cov(\mathbf{R}_t) = \boldsymbol{\Omega}_{FM} = \mathbf{B}\boldsymbol{\Omega}_f\mathbf{B}' + \mathbf{D} \tag{4}$$

Note, (4) implies that

$$\begin{aligned} var(R_{it}) &= \boldsymbol{\beta}_i'\boldsymbol{\Omega}_f\boldsymbol{\beta}_i + \sigma_i^2 \\ cov(R_{it}, R_{jt}) &= \boldsymbol{\beta}_i'\boldsymbol{\Omega}_f\boldsymbol{\beta}_j \end{aligned}$$

Portfolio Analysis

Let $\mathbf{w} = (w_1, \ldots, w_n)$ be a vector of portfolio weights (w_i = fraction of wealth in asset i). If $\mathbf{R}_t$ is the $(N \times 1)$ vector of simple returns then

$$R_{p,t} = \mathbf{w}'\mathbf{R}_t = \sum_{i=1}^{N} w_i R_{it}$$

Portfolio Factor Model

$$\begin{aligned}
\mathbf{R}_t &= \boldsymbol{\alpha} + \mathbf{B}\mathbf{f}_t + \boldsymbol{\varepsilon}_t \Rightarrow \\
R_{p,t} &= \mathbf{w}'\boldsymbol{\alpha} + \mathbf{w}'\mathbf{B}\mathbf{f}_t + \mathbf{w}'\boldsymbol{\varepsilon}_t = \alpha_p + \boldsymbol{\beta}_p'\mathbf{f}_t + \varepsilon_{p,t} \\
\alpha_p &= \mathbf{w}'\boldsymbol{\alpha},\ \boldsymbol{\beta}_p' = \mathbf{w}'\mathbf{B},\ \varepsilon_{p,t} = \mathbf{w}'\boldsymbol{\varepsilon}_t \\
var(R_{p,t}) &= \boldsymbol{\beta}_p'\boldsymbol{\Omega}_f\boldsymbol{\beta}_p + var(\varepsilon_{p,t}) = \mathbf{w}'\mathbf{B}\boldsymbol{\Omega}_f\mathbf{B}'\mathbf{w} + \mathbf{w}'\mathbf{D}\mathbf{w}
\end{aligned}$$

Macroeconomic Factor Models

$$\begin{aligned} R_{it} &= \alpha_i + \boldsymbol{\beta}_i' \mathbf{f}_t + \varepsilon_{it} \\ \mathbf{f}_t &= \text{observed economic/financial time series} \end{aligned}$$

Econometric problems:

- Choice of factors
- Estimate factor betas, $\boldsymbol{\beta}_i$, and residual variances, σ_i^2, using time series regression techniques.
- Estimate factor covariance matrix, $\boldsymbol{\Omega}_f$, from observed history of factors

Risk Measures

Let R_t be an iid random variable, representing the return on an asset at time t, with pdf f, cdf F, $E[R_t] = \mu$ and $var(R_t) = \sigma^2$.

The most common risk measures associated with R_t are

1. Return standard deviation: $\sigma = SD(R_t) = \sqrt{var(R_t)}$

2. Value-at-Risk: $VaR_\alpha = q_\alpha = F^{-1}(\alpha)$, $\alpha \in (0.01, 0.10)$

3. Expected tail loss: $ETL_\alpha = E[R_t | R_t \leq VaR_\alpha]$, $\alpha \in (0.01, 0.10)$

Note: VaR_α and ETL_α are *tail-risk* measures.

Risk Measures: Normal Distribution

$$R_t \sim iid\ N(\mu, \sigma^2)$$
$$\begin{aligned} R_t &= \mu + \sigma \times Z,\ Z \sim iid\ N(0,1) \\ \Phi &= F_Z,\ \phi = f_Z \end{aligned}$$

Value-at-Risk

$$VaR_\alpha^N = \mu + \sigma \times z_\alpha,\ z_\alpha = \Phi^{-1}(\alpha)$$

Expected tail loss

$$ETL_\alpha^N = \mu - \sigma \frac{1}{\alpha} \phi(z_\alpha)$$

Estimation

$$
\begin{aligned}
\hat{\mu} &= \frac{1}{T}\sum_{t=1}^{T} R_t,\ \hat{\sigma}^2 = \frac{1}{T-1}\sum_{t=1}^{T}(R_t - \hat{\mu})^2,\ \hat{\sigma} = \sqrt{\hat{\sigma}^2} \\
\widehat{VaR}_{\alpha}^{N} &= \hat{\mu} + \hat{\sigma} \times z_{\alpha} \\
\widehat{ETL}_{\alpha}^{N} &= \hat{\mu} - \hat{\sigma}\frac{1}{\alpha}\phi(z_{\alpha})
\end{aligned}
$$

Note: Standard errors are rarely reported for $\widehat{VaR}_{\alpha}^{N}$ and $\widehat{ETL}_{\alpha}^{N}$, but are easy to compute using “delta method” or bootstrap.

Risk Measures: Factor Model and Normal Distribution

$$R_t = \alpha + \boldsymbol{\beta}'\mathbf{f}_t + \varepsilon_t$$

$$\mathbf{f}_t \sim iid\ N(\boldsymbol{\mu}_f, \boldsymbol{\Omega}_f),\ \varepsilon_t \sim iid\ N(0, \sigma_\varepsilon^2),\ cov(f_{k,t}, \varepsilon_s) = 0 \text{ for all } k, t, s$$

Then

$$\begin{aligned}
E[R_t] &= \mu_{FM} = \alpha + \boldsymbol{\beta}'\boldsymbol{\mu}_f \\
var(R_t) &= \sigma_{FM}^2 = \boldsymbol{\beta}'\boldsymbol{\Omega}_f\boldsymbol{\beta} + \sigma_\varepsilon^2 \\
\sigma_{FM} &= \sqrt{\boldsymbol{\beta}'\boldsymbol{\Omega}_f\boldsymbol{\beta} + \sigma_\varepsilon^2} \\
VaR_\alpha^{N,FM} &= \mu_{FM} + \sigma_{FM} \times z_\alpha \\
ETL_\alpha^{N,FM} &= \mu_{FM} - \sigma_{FM}\frac{1}{\alpha}\phi(z_\alpha)
\end{aligned}$$

Note: In practice, $\alpha = 0$ is typically imposed so that $\mu_{FM} = \boldsymbol{\beta}'\boldsymbol{\mu}_f$.

Tail Risk Measures: Non-Normal Distributions

Stylized fact: The empirical distribution of many asset returns exhibit asymmetry and fat tails

Some commonly used non-normal distributions for

- Skewed Student's t (fat-tailed and asymmetric)
- Generalized hyperbolic
- Cornish-Fisher Approximations
- Extreme value theory: Generalized Pareto

Modeling Non-Normal Returns for VaR Calculations

$$R_t = \mu + \sigma Z_t,$$
$$E[R_t] = \mu,\ var(R_t) = \sigma^2$$
$$Z_t \sim iid\ (0,1) \text{ with } CDF\ F_Z$$

Then

$$VaR_q = F^{-1}(q) = \mu + \sigma \cdot F_Z^{-1}(q)$$

$$\begin{aligned} \text{normal VaR:} & \quad F_Z^{-1}(q) = \text{N(0,1) quantile} \\ \text{Student's t VaR} & : \quad F_Z^{-1}(q) = \text{Student's t quantile} \\ \text{Cornish-Fisher (modified) VaR} & : \quad F_Z^{-1}(q) = \text{Cornish-Fisher quantile} \\ \text{EVT VaR} & : \quad F_Z^{-1}(q) = \text{GPD quantile} \end{aligned}$$

Tail Risk Measures: Cornish-Fisher Approximation

Idea: Approximate unknown CDF of $Z = (R - \mu)/\sigma$ using 2 term Edgeworth expansion around normal CDF $\Phi(\cdot)$ and invert expansion to get quantile estimate:

$$\begin{aligned} F^{-1}_{Z,CF}(q) &= z_q + \frac{1}{6}(z_q^2 - 1) \times skew + \frac{1}{24}(z_q^3 - 3z_q) \times ekurt \\ &\quad - \frac{1}{36}(2z_q^3 - 5z_q) \times skew \\ z_q &= \Phi^{-1}(q),\ skew = E[Z^3],\ ekurt = E[Z^4] \end{aligned}$$

Note: Very commonly used in industry

Reference: Boudt, Peterson and Croux (2008) "Estimation and Decomposition of Downside Risk for Portfolios with Nonnormal Returns," *Journal of Risk.*

Tail Risk Measures: Non-parametric estimates

Assume R_t is iid but make no distributional assumptions:

$$\{R_1, \ldots, R_T\} = \text{observed iid sample}$$

Estimate risk measures using sample statistics (aka *historical simulation*)

$$\begin{aligned}
\widehat{VaR}_{\alpha}^{HS} &= \hat{q}_{\alpha} = \text{ empirical } \alpha - \text{quantile} \\
\widehat{ETL}_{\alpha}^{HS} &= \frac{1}{[T\alpha]} \sum_{t=1}^{T} R_t \cdot 1\{R_t \leq \hat{q}_{\alpha}\} \\
1\{R_t \leq \hat{q}_{\alpha}\} &= 1 \text{ if } R_t \leq \hat{q}_{\alpha};\ 0 \text{ otherwise}
\end{aligned}$$

Factor Risk Budgeting

- Additively decompose (slice and dice) individual asset or portfolio return risk measures into factor contributions

- Allow portfolio manager to know sources of factor risk for allocation and hedging purposes

- Allow risk manager to evaluate portfolio from factor risk perspective

Factor Risk Decompositions

Assume asset or portfolio return R_t can be explained by a factor model

$$R_t = \alpha + \boldsymbol{\beta}'\mathbf{f}_t + \varepsilon_t$$
$$\mathbf{f}_t \sim iid\ (\boldsymbol{\mu}_f, \boldsymbol{\Omega}_f),\ \varepsilon_t \sim iid\ (0, \sigma_\varepsilon^2),\ cov(f_{k,t}, \varepsilon_s) = 0 \text{ for all } k, t, s$$

Re-write the factor model as

$$\begin{aligned}
R_t &= \alpha + \boldsymbol{\beta}'\mathbf{f}_t + \varepsilon_t = \alpha + \boldsymbol{\beta}'\mathbf{f}_t + \sigma_\varepsilon \times z_t \\
&= \alpha + \tilde{\boldsymbol{\beta}}'\tilde{\mathbf{f}}_t \\
\tilde{\boldsymbol{\beta}} &= (\boldsymbol{\beta}', \sigma_\varepsilon)',\ \tilde{\mathbf{f}}_t = (\mathbf{f}_t, z_t)',\ z_t = \frac{\varepsilon_t}{\sigma_\varepsilon} \sim iid\ (0, 1)
\end{aligned}$$

Then

$$\sigma_{FM}^2 = \tilde{\boldsymbol{\beta}}'\boldsymbol{\Omega}_{\tilde{f}}\tilde{\boldsymbol{\beta}},\ \boldsymbol{\Omega}_{\tilde{f}} = \begin{pmatrix} \boldsymbol{\Omega}_f & 0 \\ 0 & 1 \end{pmatrix}$$

Linearly Homogenous Risk Functions

Let $RM(\tilde{\boldsymbol{\beta}})$ denote the risk measures σ_{FM}, VaR_{α}^{FM} and ETL_{α}^{FM} as functions of $\tilde{\boldsymbol{\beta}}$

Result 1: $RM(\tilde{\boldsymbol{\beta}})$ is a linearly homogenous function of $\tilde{\boldsymbol{\beta}}$ for $RM = \sigma_{FM}$, VaR_{α}^{FM} and ETL_{α}^{FM}. That is, $RM(c \cdot \tilde{\boldsymbol{\beta}}) = c \cdot RM(\tilde{\boldsymbol{\beta}})$ for any constant $c \geq 0$

Example: Consider $RM(\tilde{\boldsymbol{\beta}}) = \sigma_{FM}(\tilde{\boldsymbol{\beta}})$. Then

$$\begin{aligned}\sigma_{FM}(c \cdot \tilde{\boldsymbol{\beta}}) &= \left(c \cdot \tilde{\boldsymbol{\beta}}' \boldsymbol{\Omega}_{\tilde{f}} c \cdot \tilde{\boldsymbol{\beta}}\right)^{1/2} = c \cdot \left(\tilde{\boldsymbol{\beta}}' \boldsymbol{\Omega}_{\tilde{f}} \tilde{\boldsymbol{\beta}}\right)^{1/2} \\ &= c \cdot \sigma_{FM}(\tilde{\boldsymbol{\beta}})\end{aligned}$$

Euler's Theorem and Additive Risk Decompositions

Result 2: Because $RM(\tilde{\boldsymbol{\beta}})$ is a linearly homogenous function of $\tilde{\boldsymbol{\beta}}$, by Euler's Theorem

$$\begin{aligned}
RM(\tilde{\boldsymbol{\beta}}) &= \sum_{j=1}^{k+1} \tilde{\beta}_j \frac{\partial RM(\tilde{\boldsymbol{\beta}})}{\partial \tilde{\beta}_j} \\
&= \tilde{\beta}_1 \frac{\partial RM(\tilde{\boldsymbol{\beta}})}{\partial \tilde{\beta}_1} + \cdots + \tilde{\beta}_{k+1} \frac{\partial RM(\tilde{\boldsymbol{\beta}})}{\partial \tilde{\beta}_{k+1}} \\
&= \beta_1 \frac{\partial RM(\tilde{\boldsymbol{\beta}})}{\partial \beta_1} + \cdots + \beta_k \frac{\partial RM(\tilde{\boldsymbol{\beta}})}{\partial \beta_k} + \sigma_\varepsilon \frac{\partial RM(\tilde{\boldsymbol{\beta}})}{\partial \sigma_\varepsilon}
\end{aligned}$$

Terminology

Factor j *marginal contribution to risk*

$$\frac{\partial RM(\tilde{\boldsymbol{\beta}})}{\partial \tilde{\beta}_j}$$

Factor j *contribution to risk*

$$\tilde{\beta}_j \frac{\partial RM(\tilde{\boldsymbol{\beta}})}{\partial \tilde{\beta}_j}$$

Factor j *percent contribution to risk*

$$\frac{\tilde{\beta}_j \frac{\partial RM(\tilde{\boldsymbol{\beta}})}{\partial \tilde{\beta}_j}}{RM(\tilde{\boldsymbol{\beta}})}$$

Analytic Results for $RM(\tilde{\boldsymbol{\beta}}) = \sigma_{FM}(\tilde{\boldsymbol{\beta}})$

$$\begin{aligned} \sigma_{FM}(\tilde{\boldsymbol{\beta}}) &= \left(\tilde{\boldsymbol{\beta}}'\boldsymbol{\Omega}_{\tilde{f}}\tilde{\boldsymbol{\beta}}\right)^{1/2} \\ \frac{\partial \sigma_{FM}(\tilde{\boldsymbol{\beta}})}{\partial \tilde{\boldsymbol{\beta}}} &= \frac{1}{\sigma_{FM}(\tilde{\boldsymbol{\beta}})}\boldsymbol{\Omega}_{\tilde{f}}\tilde{\boldsymbol{\beta}} \end{aligned}$$

Factor $j = 1, \ldots, K$ percent contribution to $\sigma_{FM}(\tilde{\boldsymbol{\beta}})$

$$\frac{\beta_1 \beta_j cov(f_{1t}, f_{jt}) + \cdots + \beta_j^2 var(f_{jt}) + \cdots + \beta_K \beta_j cov(f_{Kt}, f_{jt})}{\sigma_{FM}^2(\tilde{\boldsymbol{\beta}})},$$

Asset specific factor contribution to risk

$$\frac{\sigma_{\varepsilon}^2}{\sigma_{FM}^2(\tilde{\boldsymbol{\beta}})},\ j = K + 1$$

Results for $RM(\tilde{\boldsymbol{\beta}}) = VaR_\alpha^{FM}(\tilde{\boldsymbol{\beta}})$, $ETL_\alpha^{FM}(\tilde{\boldsymbol{\beta}})$

Based on arguments in Scaillet (2002), Meucci (2007) showed that

$$\begin{aligned}
\frac{\partial VaR_\alpha^{FM}(\tilde{\boldsymbol{\beta}})}{\partial \tilde{\beta}_j} &= E[\tilde{f}_{jt} | R_t = VaR_\alpha^{FM}(\tilde{\boldsymbol{\beta}})],\ j = 1, \ldots, K+1 \\
\frac{\partial ETL_\alpha^{FM}(\tilde{\boldsymbol{\beta}})}{\partial \tilde{\beta}_j} &= E[\tilde{f}_{jt} | R_t \leq VaR_\alpha^{FM}(\tilde{\boldsymbol{\beta}})],\ j = 1, \ldots, K+1
\end{aligned}$$

Remarks

- Intuitive interpretation as stress loss scenario
- Analytic results are available under normality

Marginal Contributions to Tail Risk: Non-Parametric Estimates

Assume R_t and $\tilde{\mathbf{f}}_t$ are iid but make no distributional assumptions:

$$\{(R_1, \tilde{\mathbf{f}}_1), \ldots, (R_T, \tilde{\mathbf{f}}_T)\} = \text{observed iid sample}$$

Estimate marginal contributions to risk using *historical simulation*

$$\begin{aligned} \hat{E}^{HS}[\tilde{f}_{jt}|R_t &= VaR_\alpha] = \\ &\frac{1}{m}\sum_{t=1}^{T} \tilde{f}_{jt} \cdot 1\left\{\widehat{VaR}_\alpha^{HS} - \varepsilon \leq R_t \leq \widehat{VaR}_\alpha^{HS} + \varepsilon\right\} \\ \hat{E}^{HS}[\tilde{f}_{jt}|R_t &\leq VaR_\alpha] = \frac{1}{[T\alpha]}\sum_{t=1}^{T} \tilde{f}_{jt} \cdot 1\left\{\widehat{VaR}_\alpha^{HS} \leq R_t\right\} \end{aligned}$$

Problem: Not reliable with small samples or with unequal histories for R_t

Portfolio Risk Budgeting

- Additively decompose (slice and dice) portfolio risk measures into asset contributions

- Allow portfolio manager to know sources of asset risk for allocation and hedging purposes

- Allow risk manager to evaluate portfolio from asset risk perspective

Portfolio Risk Decompositions

Portfolio return:

$$\begin{aligned} \mathbf{R}_t &= (R_{1t}, \ldots, R_{Nt}),\ \mathbf{w}_t = (w_1, \ldots, w_n)' \\ R_{p,t} &= \mathbf{w}'\mathbf{R}_t = \sum_{i=1}^{N} w_i R_{it} \end{aligned}$$

Let $RM(\mathbf{w})$ denote the risk measures σ, VaR_α and ETL_α as functions of the portfolio weights $\mathbf{w}$.

Result 3: $RM(\mathbf{w})$ is a linearly homogenous function of $\mathbf{w}$ for $RM = \sigma$, VaR_α and ETL_α. That is, $RM(c \cdot \mathbf{w}) = c \cdot RM(\mathbf{w})$ for any constant $c \geq 0$

Result 4: Because $RM(\mathbf{w})$ is a linearly homogenous function of $\mathbf{w}$, by Euler's Theorem

$$\begin{aligned} RM(\mathbf{w}) &= \sum_{i=1}^{N} w_i \frac{\partial RM(\mathbf{w})}{\partial w_i} \\ &= w_1 \frac{\partial RM(\mathbf{w})}{\partial w_1} + \cdots + w_N \frac{\partial RM(\mathbf{w})}{\partial w_N} \end{aligned}$$

Terminology

Asset i *marginal contribution to risk*

$$\frac{\partial RM(\mathbf{w})}{\partial w_i}$$

Asset i *contribution to risk*

$$w_i \frac{\partial RM(\mathbf{w})}{\partial w_i}$$

Asset i *percent contribution to risk*

$$\frac{w_i \frac{\partial RM(\mathbf{w})}{\partial w_i}}{RM(\mathbf{w})}$$

Analytic Results for $RM(\mathbf{w}) = \sigma(\mathbf{w})$

$$
\begin{aligned}
R_{p,t} &= \mathbf{w}'\mathbf{R}_t,\ var(\mathbf{R}_t) = \mathbf{\Omega} \\
\sigma(\mathbf{w}) &= \left(\mathbf{w}'\mathbf{\Omega}\mathbf{w}\right)^{1/2} \\
\frac{\partial \sigma(\mathbf{w})}{\partial \mathbf{w}} &= \frac{1}{\sigma(\mathbf{w})}\mathbf{\Omega}\mathbf{w}
\end{aligned}
$$

Note

$$
\begin{aligned}
\mathbf{\Omega}\mathbf{w} &= \begin{pmatrix} cov(R_{1t}, R_{p,t}) \\ \vdots \\ cov(R_{Nt}, R_{p,t}) \end{pmatrix} = \sigma\left(\mathbf{w}\right) \begin{pmatrix} \beta_{1,p} \\ \vdots \\ \beta_{N,p} \end{pmatrix} \\
\beta_{i,p} &= cov(R_{it}, R_{p,t})/\sigma^2\left(\mathbf{w}\right)
\end{aligned}
$$

Results for $RM(\mathbf{w}) = VaR_\alpha(\mathbf{w})$, $ETL_\alpha(\mathbf{w})$

Gourieroux (2000) et al and Scalliet (2002) showed that

$$\begin{aligned}
\frac{\partial VaR_\alpha(\mathbf{w})}{\partial w_i} &= E[R_{it} | R_{p,t} = VaR_\alpha(\mathbf{w})],\ i = 1, \ldots, N \\
\frac{\partial ETL_\alpha(\mathbf{w})}{\partial w_i} &= E[R_{it} | R_{p,t} \leq VaR_\alpha(\mathbf{w})],\ i = 1, \ldots, N
\end{aligned}$$

Remarks

- Intuitive interpretation as stress loss scenario
- Analytic results are available under normality and Cornish-Fisher expansion

Marginal Contributions to Tail Risk: Non-Parametric Estimates

Assume the $N \times 1$ vector of returns $\mathbf{R}_t$ is iid but make no distributional assumptions:

$$\begin{aligned} \{\mathbf{R}_t, \ldots, \mathbf{R}_T\} &= \text{observed iid sample} \\ R_{p,t} &= \mathbf{w}'\mathbf{R}_t \end{aligned}$$

Estimate marginal contributions to risk using *historical simulation*

$$\begin{aligned} \hat{E}^{HS}[R_{it}|R_{p,t} &= VaR_\alpha] = \\ & \frac{1}{m}\sum_{t=1}^{T} R_{it} \cdot 1\left\{\widehat{VaR}_\alpha^{HS} - \varepsilon \leq R_{p,t} \leq \widehat{VaR}_\alpha^{HS} + \varepsilon\right\} \\ \hat{E}^{HS}[R_{it}|R_{p,t} &\leq VaR_\alpha] = \frac{1}{[T\alpha]}\sum_{t=1}^{T} R_{it} \cdot 1\left\{\widehat{VaR}_\alpha^{HS} \leq R_{p,t}\right\} \end{aligned}$$

Problem: Very few observations used for estimates

Marginal Contributions to Tail Risk: Cornish-Fisher Expansion

Boudt, Peterson and Croux (2008) derived analytic expressions for

$$\begin{aligned}\frac{\partial VaR_{\alpha}(\mathbf{w})}{\partial w_i} &= E[R_{it}|R_{p,t} = VaR_{\alpha}(\mathbf{w})],\ i = 1, \ldots, N \\ \frac{\partial ETL_{\alpha}(\mathbf{w})}{\partial w_i} &= E[R_{it}|R_{p,t} \leq VaR_{\alpha}(\mathbf{w})],\ i = 1, \ldots, N\end{aligned}$$

based on the Cornish-Fisher quantile expansions for each asset.

- Results depend on asset specific variance, skewness, kurtosis as well as all pairwise covariances, co-skewnesses and co-kurtosises

Factor Model Monte Carlo

Problem: Short history and incomplete data limits applicability of historical simulation, and risk budgeting calculations are extremely difficult for non-normal distributions

Solution: *Factor Model Monte Carlo (FMMC)*

- Use fitted factor model to simulate pseudo hedge fund return data preserving empirical characteristics
  - Use full history for factors and observed history for asset returns
  - Do not assume full parametric distributions for hedge fund returns and risk factor returns

- Estimate tail risk and related measures non-parametrically from simulated return data

Unequal History

$$\begin{array}{cccc} f_{1T} & \cdots & f_{KT} & R_{iT} \\ \vdots & \vdots & \vdots & \vdots \\ f_{1,T-T_i+1} & \cdots & f_{1,T-T_i+1} & R_{i,T-T_i+1} \\ \vdots & \vdots & \vdots & \\ f_{11} & \cdots & f_{1K} & \end{array}$$

- Observe full history for factors $\{\mathbf{f}_1, \ldots, \mathbf{f}_T\}$

- Observe partial history for assets (monotone missing data)

$$\{R_{i,T-T_i+1}, \ldots, R_{iT}\},$$
$$i = 1, \ldots, n;\ t = T - T_i + 1, \ldots, T$$

Simulation Algorithm

- Estimate factor models for each asset using partial history for assets and risk factors

$$R_{it} = \hat{\alpha}_i + \hat{\boldsymbol{\beta}}_i' \mathbf{f}_t + \hat{\varepsilon}_{it},\ t = T - T_i + \mathbf{1}, \ldots, T$$

- Simulate B values of the risk factors by re-sampling with replacement from *full history* of risk factors $\{\mathbf{f}_1, \ldots, \mathbf{f}_T\}$:

$$\{\mathbf{f}_1^*, \ldots, \mathbf{f}_B^*\}$$

- Simulate B values of the factor model residuals from empirical distribution or fitted non-normal distribution:

$$\{\hat{\varepsilon}_{i1}^*, \ldots, \hat{\varepsilon}_{iB}^*\}$$

- Create pseudo factor model returns from fitted factor model parameters, simulated factor variables and simulated residuals:

$$\{R_1^*, \ldots, R_B^*\}$$
$$R_{it}^* = \hat{\boldsymbol{\beta}}_i' \mathbf{f}_t^* + \hat{\varepsilon}_{it}^*, \ t = 1, \ldots, B$$

Remarks:

1. Algorithm does not assume normality, but relies on linear factor structure for distribution of returns given factors.

2. Under normality (for risk factors and residuals), FMMC algorithm reduces to MLE with monotone missing data.

3. Use of full history of factors is key for improved efficiency over truncated sample analysis

4. Technical justification is detailed in Jiang (2009).

Simulating Factor Realizations: Choices

- Empirical distribution
- Filtered historical simulation
  - use local time-varying factor covariance matrices to standardize factors prior to re-sampling and then re-transform with covariance matrices after re-sampling
- Multivariate non-normal distributions

Simulating Residuals: Distribution choices

- Empirical
- Normal
- Skewed Student's t
- Generalized hyperbolic
- Cornish-Fisher

Reverse Optimization, Implied Returns and Tail Risk Budgeting

- Standard portfolio optimization begins with a set of expected returns and risk forecasts.

- These inputs are fed into an optimization routine, which then produces the portfolio weights that maximizes some risk-to-reward ratio (typically subject to some constraints).

- Reverse optimization, by contrast, begins with a set of portfolio weights and risk forecasts, and then infers what the implied expected returns must be to satisfy optimality.

Optimized Portfolios

Suppose that the objective is to form a portfolio by maximizing a generalized expected return-to-risk (Sharpe) ratio:

$$\begin{aligned} &\max_{\mathbf{w}} \frac{\mu_p(\mathbf{w})}{RM(\mathbf{w})} \\ \mu_p(\mathbf{w}) &= \mathbf{w}'\boldsymbol{\mu} \\ RM(\mathbf{w}) &= \text{linearly homogenous risk measure} \end{aligned}$$

The F.O.C.'s of the optimization are $(i = 1, \ldots, n)$

$$0 = \frac{\partial}{\partial w_i}\left(\frac{\mu_p(\mathbf{w})}{RM(\mathbf{w})}\right) = \frac{1}{RM(\mathbf{w})}\frac{\partial \mu_p(\mathbf{w})}{\partial w_i} - \frac{\mu_p(\mathbf{w})}{RM(\mathbf{w})^2}\frac{\partial RM(\mathbf{w})}{\partial w_i}$$

Reverse Optimization and Implied Returns

Reverse optimization uses the above optimality condition with fixed portfolio weights to determine the optimal fund expected returns. These optimal expected returns are called *implied returns*. The implied returns satisfy

$$\mu_i^{\text{implied}}(\mathbf{w}) = \frac{\mu_p(\mathbf{w})}{RM(\mathbf{w})} \times \frac{\partial RM(\mathbf{w})}{\partial w_i}$$

Result: fund i's implied return is proportional to its marginal contribution to risk, with the constant of proportionality being the generalized Sharpe ratio of the portfolio.

How to Use Implied Returns

- For a given generalized portfolio Sharpe ratio, $\mu_i^{\text{implied}}(\mathbf{w})$ is large if $\frac{\partial RM(\mathbf{w})}{\partial w_i}$ is large.

- If the actual or forecast expected return for fund i is less than its implied return, then one should reduce one's holdings of that asset

- If the actual or forecast expected return for fund i is greater than its implied return, then one should increase one's holdings of that asset

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