The Relationship between the Beveridge-Nelson Decomposition and Unobserved Components Models with Correlated Shocks

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Abstract

The Beveridge-Nelson (BN) decomposition is a model-based method for decomposing time series into permanent and transitory components. It is closely related to decompositions based on unobserved components (UC) models with random walk trends and covariance stationary cycles. The decomposition when extended to I(2) models can also be related to non-model based signal extraction filters such as the HP filter. We show that the BN decomposition provides information on the correlation between the permanent and transitory shocks in a certain class of UC models. The correlation between components is known to determine the smoothed estimates of components from UC models. The BN decomposition can also be used to evaluate the efficacy of alternative methods. We also show, contrary to popular belief, that the BN decomposition can produce smooth cycles if the reduced form forecasting model is appropriately specified.

1 Introduction

The Beveridge-Nelson (BN) decomposition is a model-based method for decomposing a univariate or multivariate time series into permanent and transitory (PT) components. It begins with a definition of the stochastic trend as the limiting forecast of the level of the series minus any deterministic components given the current information set. The permanent component is a pure random walk while the remaining movements in the series are the I(0) transitory component. Other than the random walk trend, the BN decomposition does not make assumptions about the structure of the components and the correlations between them. However, it is closely related to decompositions based on unobserved components (UC) models with random walk trends and covariance stationary cycles. The BN decomposition can also be related to non-model based signal extraction filters such as the Hodrick-Prescott (HP) filter and other Butterworth lowpass filters considered by Gomez (2001). These latter methods are indirectly related to the BN decomposition through their relationships with UC models.

Our contribution in this paper is to clarify the relationship between the BN decomposition and other univariate detrending methods popular in economics. In particular, for certain I(1) and I(2) models we show the relationship between the BN decomposition and UC models with correlated permanent and transitory shocks and we clarify when the correlation between shocks is identified. Furthermore, for a particular class of UC models, we show how the ARIMA model used to compute the BN decomposition can be used to determine the range of correlation values such that the real-time trend and cycle estimates from the UC models are equivalent to the BN decomposition. We also demonstrate how the BN decomposition can be used as a benchmark to test the over-identifying restrictions that are commonly made in applied macroeconomics research. Examining the over-identifying restrictions can help applied researchers understand some of the assumptions they make and the resulting trade-offs that exist. We emphasize that smoothed estimates from UC models can potentially be unidentified. If the random walk trend is the correct model, restrictions placed on the parameter space that are commonly made in the literature can result in spurious cycles. Many of these results have been stated previously in the literature and part of our contribution is to present these results in a cohesive manner.

The BN decomposition holds less relevance for researchers who believe that the trend is not a pure random walk. Consequently, our analysis is limited to models with random walk trend components. There exist other types of PT decompositions in which the permanent component is an integrated series but not a pure random walk. These include the canonical decomposition of Hillmer and Tiao (1982) and the general PT decompositions of Quah (1992), but these are outside the scope of this paper. As emphasized by Quah (1992), the random walk trend implicit in the BN decomposition maximizes the importance of the permanent component. This should always be recognized when interpreting the results of the BN decomposition.

2 The BN Decomposition of an I(1) Process

Assume that the univariate time series y_t is an $I(1)$ process with Wold representation given by

$$\Delta y_t = \mu + \psi(L)\epsilon_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j} \tag{1}$$

where $\Delta = 1 - L$, $\psi(0) = 1, \psi(1) \neq 0$, $\sum_{j=0}^{\infty} j^{1/2}|\psi_j| < \infty$, and ϵ_t are iid $(0, \sigma^2)$ one-step-ahead forecast errors. The permanent component or trend τ_t of the BN decomposition of y_t is defined as the limiting forecast minus any deterministic components

$$\tau_t^{BN} = \lim_{J \to \infty} E[y_{t+J} - J\mu|\Omega_t] \tag{2}$$

Writing $y_{t+J} = y_t + \Delta y_{t+1} + \cdots + \Delta y_{t+J}$ and using $E[\Delta y_{t+j}|\Omega_t]$ $(j = 1, \ldots, J)$ based on (1) allows for the analytic evaluation of τ_t^{BN} as

$$\tau_t^{BN} = \mu + \tau_{t-1}^{BN} + \psi(1)\epsilon_t \tag{3}$$

Hence, the BN trend is a pure random walk with drift μ and has innovation variance $\sigma^2\psi(1)^2$. The transitory component or cycle, c_t^{BN}, is defined as the difference between y_t and the BN trend

$$c_t^{BN} = y_t - \tau_t^{BN} = \tilde{\psi}(L)\epsilon_t \tag{4}$$

where $\tilde{\psi}(L) = \sum_{j=0}^{\infty} \tilde{\psi}_j L^j$ and $\tilde{\psi}_j = -\sum_{k=j+1}^{\infty} \psi_k$. Solo (1989) showed that the $\frac{1}{2}$-summability of $\psi(L)$ and the uniqueness of the Wold decomposition guarantees the existence and uniqueness of the BN decomposition. From (3) and (4) it is clear that the BN decomposition produces real-time or one-sided estimates of the permanent and transitory components at time t.

An alternative derivation of the BN decomposition follows directly from the factorization $\psi(L) = \psi(1) + (1-L)\tilde{\psi}(L)$. Then (1) may be rewritten as

$$\Delta y_t = \mu + \psi(1)\epsilon_t + (1-L)\tilde{\psi}(L)\epsilon_t \tag{5}$$

which identifies $(\mu + \psi(1)\epsilon_t)/(1-L)$ as the permanent component and $\tilde{\psi}(L)\epsilon_t$ as the transitory component.

In practice, the BN decomposition may be computed in a number of ways. Typically, it is assumed that $\psi(L) = \theta(L)/\phi(L)$ where the orders of $\phi(L)$ and $\theta(L)$ are p and q, respectively, and the roots of $\phi(L) = 0$ and $\theta(L) = 0$ are assumed to lie outside the complex unit circle. A brute force approach is based on estimating an ARMA(p,q) model for Δy_t, using these estimates to compute an estimate of $\psi(1) = \theta(1)/\phi(1)$, and then forming estimates of the components using (2) and (4) with the ARMA residuals in place of ϵ_t. Cuddington and Winters (1987), Miller (1988) and Newbold (1990) provided improvements to this brute force method. These methods are valid if the forecasting model for Δy_t is a univariate ARMA(p,q) model. Ariño and Newbold (1998) extended the algorithm of Newbold (1990) to multivariate forecasting models for Δy_t. See also Evans and Reichlin (1994) for a discussion of the BN decomposition for multivariate models. Recently, Morley (2002) provided a very simple state-space approach for calculating the BN decomposition that is valid for any forecasting model for Δy_t that can be cast into state-space form. In particular, suppose $\Delta y_t - \mu$ is a linear combination of the elements of the $m \times 1$ state vector $\boldsymbol{\alpha}_t$

$$\Delta y_t - \mu = \mathbf{z}'\boldsymbol{\alpha}_t$$

where $\mathbf{z}$ is an $m \times 1$ vector with fixed elements. Suppose further that

$$\boldsymbol{\alpha}_t = \mathbf{T}\boldsymbol{\alpha}_{t-1} + \boldsymbol{\eta}_t,\ \ \boldsymbol{\eta}_t \sim \text{iid } N(\mathbf{0}, \mathbf{Q}), \tag{6}$$

such that all of the eigenvalues of $\mathbf{T}$ have modulus less than unity, and $\mathbf{I}_m - \mathbf{T}$ is invertible. Then, Morley

(2002) showed that

$$
\begin{aligned}
\tau_t^{BN} &= y_t + \mathbf{z}'\mathbf{T}(\mathbf{I}_m - \mathbf{T})^{-1}\boldsymbol{\alpha}_{t|t} \\
c_t^{BN} &= y_t - \tau_t^{BN} = -\mathbf{z}'\mathbf{T}(\mathbf{I}_m - \mathbf{T})^{-1}\boldsymbol{\alpha}_{t|t}
\end{aligned} \tag{7}
$$

where $\boldsymbol{\alpha}_{t|t} = E[\boldsymbol{\alpha}_t|\Omega_t]$ denotes the filtered or real-time estimate of $\boldsymbol{\alpha}_t$ from the Kalman filter.[1] An important advantage of Morley's approach is its generality. It works the same way for univariate and multivariate forecasting models for Δy_t.

Disadvantages of the methods described above to compute the BN decomposition are that they lose the first observation due to differencing the data, and that they do not provide standard error bands for the extracted trend and cycle estimates. However, as discussed by Morley, Nelson, and Zivot (MNZ) (2003) and Andersen, Lo and Snyder (2006) and shown in the next section, the BN decomposition may also be computed directly using the Kalman filter from certain UC models. This allows for the use of all the data and for the calculation of standard error bands for the extracted trend and cycle estimates. It also allows for the extraction of trend and cycle estimates at time t using information in the full sample, Ω_T.

3 The BN Decomposition and Unobserved Components Models

An advantage of the BN decomposition is that it produces a decomposition into permanent and transitory components with minimal assumptions about the structure of the components. The definition of the BN trend (2) identifies the permanent component as a pure random walk, and this result can be used to link the BN decomposition with traditional UC models with random walk trends. The following subsections describe the class of UC models that are consistent with the BN decomposition. Throughout, we assume that Δy_t has a reduced form covariance stationary and invertible ARMA(p, q) representation such that $\psi(L) = \theta(L)/\phi(L)$ in (1).

3.1 Single Source of Error Model

The definitions of the BN permanent and transitory components in (3) and (4) suggest the following single-source-of-error (SSOE) state space representation[2]

$$
\begin{aligned}
y_t &= \tau_t + c_t \\
(1-L)\tau_t &= \mu + \psi(1)\ \epsilon_t \\
c_t &= \tilde{\psi}(L)\ \epsilon_t
\end{aligned} \tag{8}
$$

[1] Throughout the paper we refer to filtered estimates as real-time estimates based on information only available at time t, and smoothed estimates as final estimates based on all available sample information.

[2] Andersen, Lo and Snyder (2006) gave a slightly different, but equivalent, formulation of the SSOE model that includes ϵ_t in the measurement equation.

where $\tilde{\psi}(L)\epsilon_t \sim \text{ARMA}(p,n)$ with $n = \max(q-1,0)$. It is clear from (8) that the innovations to the permanent and transitory components are perfectly correlated

$$\rho = \frac{\text{cov}(\psi(1)\epsilon_t, \tilde{\psi}(0)\epsilon_t)}{\sqrt{\text{var}(\psi(1)\epsilon_t)\ \text{var}(\tilde{\psi}(0)\epsilon_t)}} = \frac{\psi(1)\ \tilde{\psi}(0)}{|\psi(1)\ \tilde{\psi}(0)|} = -1 \ \text{ or } \ 1$$

where the sign of ρ depends on the sign of $\tilde{\psi}(0)$. Hence, there always exists a UC representation with perfectly correlated shocks that is consistent with the BN decomposition. However, as discussed by MNZ, (8) is not the only UC representation that is consistent with the BN decomposition.

Andersen, Lo and Snyder (2006) use the SSOE representation of the BN permanent and transitory components to compute the BN decomposition directly using the Kalman filter. The filtered estimates $\tau_{t|t} = E[\tau_t|\Omega_t]$ and $c_{t|t} = E[c_t|\Omega_t]$ produced by the Kalman filter correspond to the BN permanent and transitory components (3) and (4), respectively. An advantage of this approach is that the form of the ARMA(p,n) model for $\tilde{\psi}(L)$ allows for direct calculation of $\psi(1)$. However, as noted by Watson (1986), Harvey and Koopman (2000), and emphasized in Morley (2006), the components $\tau_{t|t}$ and $c_{t|t}$ in the SSOE model are estimated with zero mean squared error because they are an exact function of past observations. As a result, the standard errors for $\tau_{t|t}$ and $c_{t|t}$ computed from the Kalman filter will be equal to zero. Morley (2006) used this result to argue against interpreting the SSOE representation as a structural model.

3.2 Two Source of Error Model

The perfect correlation between shocks to the components in (8) is due to the single disturbance term ϵ_t, which represents the forecast error in the reduced form ARMA(p,q) model for Δy_t. However, as argued by Shapiro and Watson (1988) and others, the economic forces underlying movements in real output imply multiple sources of shocks. For example, suppose that $\psi(L)\epsilon_t$ is the sum of two independent processes $\psi_1(L)\epsilon_{1t}$ and $\psi_2(L)\epsilon_{2t}$, where $\epsilon_{1t} \sim \text{iid } (0,\sigma^2_{\epsilon 1}), \epsilon_{2t} \sim \text{iid } (0,\sigma^2_{\epsilon 2})$, and $\text{cov}(\epsilon_{1t},\epsilon_{2t}) = 0$. Then equation (5) becomes

$$\begin{aligned}(1-L)y_t &= \mu + (\psi_1(1)\ \epsilon_{1t} + \psi_2(1)\ \epsilon_{2t}) + (1-L)(\tilde{\psi}_1(L)\epsilon_{1t} + \tilde{\psi}_2(L)\epsilon_{2t}) \\ &= \mu + \psi(1)\ \epsilon_t + (1-L)\left\{\tilde{\psi}_1(L)\epsilon_{1t} + \tilde{\psi}_2(L)\epsilon_{2t}\right\}\end{aligned} \tag{9}$$

The permanent innovation is still $\psi(1)\epsilon_t$ but the transitory innovation is now $\tilde{\psi}_1(0)\epsilon_{1t} + \tilde{\psi}_2(0)\epsilon_{2t}$. If $\tilde{\psi}_1(L)$ is zero,[3] then the correlation between permanent and transitory innovations will be between -1 and 1

$$\begin{aligned}\rho &= \frac{\psi_2(1)\ \tilde{\psi}_2(0)\sigma^2_{\epsilon_2}}{\sqrt{\psi_1(1)^2\ \tilde{\psi}_2(0)^2\ \sigma^2_{\epsilon_1}\sigma^2_{\epsilon_2} + \psi_2(1)^2\ \tilde{\psi}_2(0)^2\sigma^4_{\epsilon_2}}} \\ &\neq -1 \ \text{ nor } \ 1\end{aligned}$$

[3] As a simple example, let $\psi_1(L)\epsilon_t$ be a white noise process. Then, $\psi(1) = 1$ and $\tilde{\psi}_1(L) = 0$.

If the $\text{cov}(\epsilon_{1t}, \epsilon_{2t}) \neq 0$, then the correlation between the permanent and transitory shocks will be between -1 and 1 even when $\tilde{\psi}_1(L)$ is not zero.

In equation (9), the permanent component $\psi(1)\epsilon_t$ does not depend on the individual shocks. This means that the variance of the permanent component is always identified and is exactly the same as the variance of the permanent component from the SSOE model (8). In contrast, the parameters of the cycle and the correlation between the permanent and transitory shocks are generally not identified without further assumptions on the parametric form of the UC model.

To understand the general relationship between the permanent and transitory components defined by the BN decomposition from an ARMA(p, q) reduced form model for Δy_t and those defined in an unobserved components model, consider a typical UC model with two sources of shocks

$$\begin{aligned} y_t &= \tau_t + c_t \\ \tau_t &= \tau_{t-1} + d + w_t,\ w_t \sim \text{iid } (0, \sigma_w^2) \\ \phi(L)c_t &= \theta^*(L)\ v_t,\ v_t \sim \text{iid } (0, \sigma_v^2) \\ \text{cov}(w_t, v_t) &= \sigma_{wv} \end{aligned} \tag{10}$$

where the order of $\phi(L)$ is p, the order of $\theta^*(L)$ is $n = \max(q-1, 0)$ and the roots of $\phi(z) = 0$ and $\theta^*(z) = 0$ lie outside the complex unit circle. We call (10) a UC-ARMA(p, n) model. For $q > 0$, the MA order in (10) is one less than the MA order in the reduced form ARMA(p, q) model. In (10), w_t is the permanent shock and v_t is the transitory shock.

The reduced form of (10) is an ARMA model for Δy_t

$$\phi(L)(1-L)y_t = \phi(1) + \phi(L)w_t + (1-L)\theta^*(L)v_t \tag{11}$$

The MA polynomial on the right-hand side of (11) has order $\max(p, n+1)$ with respect to L[4]. As discussed in Harvey (1989), identification of the UC model parameters requires solving for these parameters uniquely from knowledge of the reduced form ARMA parameters. Since the AR polynomial in (11) and in the reduced form ARMA(p, q) model are the same, the remaining parameters to be identified from the UC model are the n MA parameters in $\theta^*(L)$ and the 3 covariance matrix parameters σ_w^2, σ_v^2, and σ_{wv}. From the MA portion of the reduced form ARMA(p, q) model, the number of moments that may be calculated is $\max(p, n+1)+1$. Therefore, the order condition for exact identification is $\max(p, n+1) + 1 = n + 3$. For example, when p is 2 and n is 0, the number of UC model parameters is equal to the number of reduced form moments and the UC model is exactly identified. MNZ used this result to estimate σ_{wv}. If $(n+3) > (\max(p, n+1) + 1)$ or $(q+1) > p$ when $q \geq 1$, the UC model (10) is under-identified. In this case, we can match the moments of

[4] A referee pointed out that if $q < p$ in the reduced form ARIMA(p, q) model, then the order of the MA component of (11) may end up being less than $\max(p, n+1)$ if w_t and v_t are perfectly correlated. For example, an ARIMA(1,1,0) reduced form model has a SSOE UC-ARMA(1,0) representation whereas a UC-ARMA(1,0) model with two sources of shocks has an ARIMA(1,1,1) reduced form.

the reduced form and the moments of the corresponding ARIMA model once we have chosen the correlation. Of course, the resulting UC model parameters must satisfy certain necessary conditions such as positive definitiveness of the covariance matrix and invertibility of the MA coefficients.

As long as a UC model with random walk trend does not restrict the parameter space which matches the moments of the observed data, the UC model produces the same filtered estimates as the BN decomposition from an unrestricted ARIMA model. For these admissible UC models, the value of the correlation between the permanent and transitory components, ρ_{wv}, does not impact the values of $\tau_{t|t}$ or $c_{t|t}$ computed by the Kalman filter. However, as noted by Harvey and Koopman (2000), Morley (2006), and Proietti (2006), the value of ρ_{wv} does impact the precision of $\tau_{t|t}$ and $c_{t|t}$ if $|\rho_{wv}| \neq 1$. In this case, $\tau_{t|t}$ and $c_{t|t}$ are estimated with non-zero mean squared error because there are now two distinct sources of error. As a result, standard error bands for the extracted components computed by the Kalman filter will be positive.

3.3 ARIMA(2,1,2) Model

To illustrate the relationship between a particular reduced form ARIMA model for y_t and the class of observationally equivalent UC-ARMA models with correlated shocks, consider the following reduced form ARIMA(2,1,2) model for y_t that was studied by MNZ

$$(1 - \phi_1 L - \phi_2 L^2)(1 - L)y_t = \mu + (1 + \theta_1 L + \theta_2 L^2)\epsilon_t \tag{12}$$

As shown in Proietti (2006), the ARIMA(2,1,2) model (12) is the unrestricted reduced form associated with the following UC-ARMA(2,1) model with correlated shocks

$$\begin{aligned} y_t &= \tau_t + c_t \\ \tau_t &= \tau_{t-1} + d + w_t, \ w_t \sim \text{iid}(0, \sigma_w^2) \\ c_t &= \phi_1 c_{t-1} + \phi_2 c_{t-2} + v_t + \theta_v v_{t-1}, \ v_t \sim \text{iid}(0, \sigma_v^2) \\ \text{cov}(w_t, v_t) &= \sigma_{wv} \end{aligned} \tag{13}$$

The reduced form implied by (13) is the following ARIMA(2,1,2) model

$$\phi(L)\Delta y_t = \phi(1)d + (w_t - \phi_1 w_{t-1} - \phi_2 w_{t-2}) + (v_t + (\theta_v - 1)v_{t-1} - \theta_v v_{t-2}) \tag{14}$$

However, not all of the parameters of the UC-ARMA(2,1) model are identified since $q+1=3>p=2$. To determine which parameters can be identified, consider the moments of the MA part of (14)

$$\begin{aligned}
\gamma_0 &= (1+\phi_1^2+\phi_2^2)\sigma_w^2 + \left(1+(\theta_v-1)^2+\theta_v^2\right)\sigma_v^2 + 2(1-\phi_1(\theta_v-1)+\phi_2\theta_v)\sigma_{wv} \\
\gamma_1 &= (-\phi_1+\phi_1\phi_2)\sigma_w^2 + ((\theta_v-1)-\theta_v(\theta_v-1))\sigma_v^2 + (-\phi_1-\phi_2(\theta_v-1) \\
&\quad +(\theta_v-1)+\theta_v\phi_1)\sigma_{wv} \\
\gamma_2 &= -\phi_2\sigma_w^2 - \theta_v\sigma_v^2 + (-\phi_2-\theta_v)\sigma_{wv}
\end{aligned} \tag{15}$$

Notice that the variance of the shock to the trend, σ_w^2, does not depend on the covariance

$$\sigma_w^2 = \frac{\gamma_0 + 2(\gamma_1+\gamma_2)}{1-2\phi_1-2\phi_2+2\phi_1\phi_2+\phi_1^2+\phi_2^2} \tag{16}$$

and is identified from the reduced form moments and parameters. The remaining parameters are not identified without further restrictions. If σ_{wv} (or ρ_{wv}) is given arbitrarily, then the remaining two unknowns (σ_v^2, θ_v) may be determined. Similarly, if σ_v^2 or θ_v is given, then (σ_{wv}, θ_v) or $(\sigma_{wv}, \sigma_v^2)$ may be determined. The system (15) is nonlinear in the parameters, however, so there may exist multiple solutions. Admissible solutions must satisfy the covariance stationarity and invertibility conditions as well as the positive definiteness of the covariance matrix of $(w_t, v_t)'$.

Assuming normal errors, all admissible solutions will have the same likelihood value as the unrestricted reduced form ARIMA(2,1,2) model and will produce the same decomposition into permanent and transitory components as the BN decomposition. This implies that there will be an observationally equivalent collection of UC-ARIMA(2,1) models, consistent with the unrestricted reduced form ARIMA(2,1,2) model, with different values of the correlation between trend and cycle innovations. This means that the reduced form ARIMA model used to compute the BN decomposition provides information about permissible correlation values between the permanent and transitory shocks in the UC model.

The SSOE model sets $\rho_{wv} = \pm 1$ in (13). Evaluating $\psi(1)$ and $\tilde{\psi}(L)$ in (8), see also Proietti and Harvey (2000), gives the resulting UC-ARMA(2,1) parameters in terms of the parameters of the unrestricted reduced form

$$\begin{aligned}
d &= \frac{1}{\phi(1)}\,\mu, \quad \sigma_w^2 = \zeta_1^2\,\sigma_\epsilon^2, \quad \sigma_v^2 = \zeta_2^2\,\sigma_\epsilon^2 \\
\theta_v &= \frac{\zeta_3}{\zeta_2}, \ \rho_{wv} = \frac{\zeta_1\zeta_2}{|\zeta_1\zeta_2|} = 1 \text{ or } -1
\end{aligned}$$

where

$$\zeta_1 = \psi(1) = \frac{\theta(1)}{\phi(1)}, \zeta_2 = \frac{\phi(1)-\theta(1)}{\phi(1)}, \zeta_3 = -\frac{\phi_2\theta(1)+\theta(2)\phi(1)}{\phi(1)}$$

As noted by Proietti (2006), the variance of the permanent component is $\psi(1)^2\sigma_\epsilon^2$, which is the same as the variance of the permanent component in the BN decomposition. Also, the sign of ρ_{wv} depends on the sign of $\phi(1) - \theta(1)$. In particular, if $\psi(1) > 1$ then $\rho_{wv} = -1$ and if $\psi(1) < 1$ then $\rho_{wv} = 1$. The SSOE model has a nonzero moving average parameter θ_v unless $\phi_2\theta(1) + \theta(2)\phi(1) = 0$, and $|\theta_v|$ can be greater than one implying a noninvertible model.

The UC-ARMA(2,1) model with correlated components considered by MNZ imposes the restriction $\theta_v = 0$ in (13). As a result, the order condition for identification is satisfied which allows σ_w^2, σ_v^2 and σ_{wv} to be recovered from (15) using

$$\begin{bmatrix} \sigma_w^2 \\ \sigma_v^2 \\ \sigma_{wv} \end{bmatrix} = \begin{bmatrix} 1+\phi_1^2+\phi_2^2 & 2 & 2(1+\phi_1) \\ -\phi_1(1-\phi_2) & -1 & -(1-\phi_2+\phi_1) \\ -\phi_2 & 0 & -\phi_2 \end{bmatrix}^{-1} \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{bmatrix}$$

Using this result, MNZ estimated ρ_{wv} to be -0.9062 for postwar quarterly real GDP. However, this is just one such UC-ARMA(2,1) model that is consistent with the reduced form. Setting $\theta_v = \theta_v^0 \neq 0$ also gives a linear system that can be inverted to obtain a solution for σ_w^2, σ_v^2,and σ_{wv}. Hence, the deduced correlation is a function of θ_v^0 and different values of θ_v^0 will produce different correlations. Therefore, to draw conclusions about the correlation between shocks in the UC-ARMA(2,1) model one must derive the set of all admissible correlation values as a function of θ_v^0. We do this for U.S. postwar quarterly real GDP in the empirical section below.

Proietti (2006) considered the UC-ARIMA(2,1) with $\rho_{wv} = 0$ and θ_v free to see if expanding the dynamics of the cycle could give a UC model with uncorrelated components that matched the moments of the US real GDP data used by MNZ. This model is also closely related to the UC model of Harvey and Jaeger (1993) in the $I(1)$ case. Unfortunately, the system of equations (15) relating the autocovariances to the remaining UC model parameters is nonlinear and it is not straightforward to determine admissibility. Proietti shows that any UC-ARMA(2,1) model with $\rho_{wv} > 0$ and $\theta_v = 0$ is equivalent to a model with $\rho_{wv} = 0$ and $\theta_v < 0$, and a model with $\rho_{wv} < 0$ and $\theta_v = 0$ is equivalent to one with $\rho_{wv} = 0$ and $\theta_v > 0$ provided σ_w/σ_v is sufficiently small.

The UC model with uncorrelated components used by Watson (1986) imposes the restrictions $\theta_v = \sigma_{wv} = 0$. From the order condition, the UC-ARMA(2,1) model is now overidentified and imposes complicated nonlinear restrictions on the reduced form ARIMA(2,1,2) model parameters. These restrictions can be checked for admissibility using the moment conditions (15). If they are not admissible, then the resulting estimated trend and cycles will be different from the BN decomposition. For example, MNZ showed that Kalman filtered trend and cycle estimates from the UC-ARMA(2,1) model with $\theta_v = \sigma_{wv} = 0$ are typically very different than those from the BN decomposition for US postwar quarterly real GDP. To explain these results, MNZ tested and firmly rejected the overidentifying restrictions using a likelihood ratio test. We note that Ord, Koehler and Snyder (1997) advocated the use of SSOE UC models because they do not impose the complicated restrictions on the reduced form that result from UC models with orthogonal components.

3.4 BN Decomposition and Smoother

In this section, we provide intuition on how the BN decomposition breaks apart a time series. We also show the effects that the correlation between components has on the smoothed estimates of the trend $\tau_{t|T} = E[\tau_t|\Omega_T]$ and cycle $c_{t|T} = E[c_t|\Omega_T]$. In a SSOE model, the components are estimated with zero mean squared error making the filtered and smoothed estimates equal to one another. This was previously noted by Watson (1986) and Harvey (1989). When the components are not perfectly correlated, the smoothed estimates will be a function of the correlation. If a unique correlation cannot be estimated from the data, smoothed estimates for the BN decomposition and any equivalent UC models are not identified without additional assumptions. Therefore, we discuss the importance of any further restrictions placed on the parameter space.

To illustrate the main issues, consider an ARIMA(0,1,1) model omitting the drift term for simplicity

$$(1-L)\,y_t = (1+\theta L)\,\epsilon_t$$

This model was also analyzed by Harvey and Koopman (2000). Figure 1 provides an example of the BN decomposition on simulated data from the ARIMA(0,1,1) process, where $\theta = 0.3$ is taken from estimates on U.S. real GDP. The circles $\bigcirc$ and the bold line connecting them are the observations. The vertical distance between two circles at two points in time is the overall shock to the series. The purpose of this graph is to highlight how the BN decomposition will break this shock into two pieces. The value of the BN trend, represented by a triangle $\triangle$, is by definition the long-run forecast of the series at time t. Using the ARIMA(0,1,1) model as the forecasting function, the forecast for Δy_t is particularly simple because the autocorrelation function of an MA(1), which determines the forecast, dies out after one period. In this case, the trend or equivalently the long run-forecast is $\tau_t = y_t + \theta\epsilon_t$. The cycle is the difference between the observation and the trend $c_t = -\theta\epsilon_t$, which is represented graphically by the vertical distances between $\bigcirc$ and $\triangle$. Notice that the cycle contains forecasting information under the assumption that the next period's innovation has expectation zero. If the cycle is negative (positive), the series is expected to move upward (downward) in the next period because it is below (above) the trend in the current period.

Figure 1 also provides intuition on how the two UC models that match the moments of the data generating process operate as well. First consider the UC model with two sources of noise. In this case, the permanent shock (ω_t) is the amount the long run forecast gets modified in a single period, $\omega_t = \tau_t - \tau_{t-1} = \triangle - \bigtriangledown$. The transitory shock ($\upsilon_t$) is the vertical distance between the observation and the trend, $\upsilon_t = \bigcirc - \triangle$. For the SSOE model, the single source of error ϵ_t will get split into ω_t and υ_t, i.e.

$$\begin{aligned}
\epsilon_t &= \Delta y_t - \theta\epsilon_{t-1} = \bigcirc - \bigtriangledown \\
\Delta y_t &= \bigcirc - \diamond \\
-\theta\epsilon_{t-1} &= \bigtriangledown - \diamond
\end{aligned}$$

Connecting the triangles ($\triangle$), we can see that the trend is not "smooth" in this example. The increased variability of the trend is caused by the positive serial correlation θ in the time series and its impact on the forecasting function, which was mentioned by Beveridge and Nelson (1981). The BN decomposition produces a highly variable trend because of the time series properties of the data. In particular, the fact that U.S. real GDP is highly persistent.

This is relevant to recent work on signal extraction and unobserved component models. Harvey and Koopman (2000) advocated using only unobserved component models whose smoothed estimates have positive and symmetric weighting patterns. Similarly, Proietti and Harvey (2000) suggested an interesting formula for a BN smoother. Both of these recommendations require that the components be uncorrelated and that the parameter space be restricted in order not to allow too high a level of persistence. Lippi and Reichlin (1992) proved that if the persistence is large i.e. $\psi(1) > 1$, then the series cannot be decomposed into uncorrelated components. Relative to the example here, a smoother with positive and symmetric weights, and consequently a model with uncorrelated components, cannot estimate this trend if it were the true model. By restricting θ from taking positive values, a smoother with symmetric weights must pass through the middle of the kinked line in Figure 1. This means that, if the restrictions on the parameter space are incorrect, the resulting decomposition would underestimate the trend and could produce cycles that are artificial and spurious.

Figure 2 includes another series simulated from the ARIMA(0,1,1) model, where we now set $\theta = -0.3$ and the random draws for ϵ_t are the same as above. This highlights the effects of the correlation on estimates of the trend. A zero correlation UC model and the BN decomposition can produce the same filtered estimates when θ is negative. Connecting the triangles to form the trend, it will pass through the middle of the kinked line in this data generating process. There exists a clear trade-off that applied researchers need to make. For real-time or filtered estimates relevant for policy analysis, the zero correlation restriction does not dramatically improve filtered estimates of the cycle while it may inhibit its estimation. This is demonstrated more clearly below. Meanwhile, the restrictions will often uniquely determine the smoothed estimates of the cycle.

4 The Relationship between the BN Decomposition and Unobserved Components Models for I(2) Processes

In the previous section, we discussed the relationship between the BN decomposition and UC models with correlated shocks for $I(1)$ processes represented by an ARIMA model. However, many empirical implementations of UC models allow the slope of the random walk trend to also evolve as a random walk, see e.g., Clark (1987) and Harvey and Jaeger (1993). This UC model allows for a more flexible trend that can pick up smooth structural breaks. In this specification, the time series y_t follows an $I(2)$ process.

For $I(1)$ processes, we stressed how the BN decomposition could be compared to UC models by comparing their reduced-form ARIMA representations. It is possible to extend this comparison for $I(2)$ models to include other popular non-model based trend/cycle decompositions. For example, Gomez (2001) demonstrated that a class of two-sided Butterworth lowpass filters and band-pass filters (built from the lowpass filters) are equivalent to UC models and consequently ARIMA models.[5] This class of nonparametric filters is based in the frequency domain and includes the Hodrick-Prescott (HP) filter as a special case. To apply these filters, a user chooses the tuning parameters of the gain function. These tuning parameters implicitly determine the underlying ARIMA model and its parameter values. Consequently, one can test the over-identifying restrictions imposed by the nonparametric filters by comparing their ARIMA representations to an unrestricted ARIMA model.

4.1 The BN Decomposition for an I(2) Process

Assume that y_t is an $I(2)$ process with a Wold representation given by

$$(1-L)^2 y_t = \psi(L)\epsilon_t, \tag{17}$$

where $\psi(L)$ and ϵ_t are defined as in (1). Using the BN factorization of $\psi(L)$, we can rewrite (17) as

$$(1-L)^2 y_t = \psi(1)\ \epsilon_t + (1-L)\tilde{\psi}(L)\epsilon_t$$

Dividing both sides by $(1-L)$ and applying the BN factorization to $\tilde{\psi}(L)$, we obtain

$$(1-L)y_t = \frac{\psi(1)}{(1-L)}\ \epsilon_t + \tilde{\psi}(1)\epsilon_t + (1-L)\tilde{\tilde{\psi}}(L)\epsilon_t \tag{18}$$

Splitting the integrated parts from the stationary part in (18), the permanent and transitory components in the BN decomposition for an $I(2)$ process may be defined as

$$(1-L)\tau_t = \frac{\psi(1)}{(1-L)}\ \epsilon_t + \tilde{\psi}(1)\epsilon_t \tag{19}$$

$$c_t = \tilde{\tilde{\psi}}(L)\epsilon_t \tag{20}$$

[5] Not all of the nonparametric filters considered by Gomez (2001) will have random walk trends, in particular the band-pass filters he considers. They consequently may not always be equivalent to the BN decomposition. Harvey and Trimbur (2003) have extended his work by building UC models with higher-order stochastic cycles that include random walk trends and slopes. These models have bandpass filter properties in terms of the cycles they can extract. Using the results in Trimbur (2006), this class of models can be shown to have reduced-form ARIMA representations and will be equivalent to the BN decomposition. Therefore, it is possible to test any over-identifying restrictions that are present, although we do not consider that here.

Defining the double integrated part of (19) as a random drift term d_t, the BN decomposition has the following SSOE UC model representation

$$\begin{aligned} y_t &= \tau_t + c_t \\ (1-L)\tau_t &= d_{t-1} + [\psi(1) + \tilde{\psi}(1)]\epsilon_t \\ (1-L)d_t &= \psi(1)\epsilon_t \\ c_t &= \tilde{\tilde{\psi}}(L)\epsilon_t \end{aligned} \tag{21}$$

Notice that in the $I(2)$ case there is an overall trend, τ_t, which follows a double random walk, a drift term, d_t, that follows a random walk, and a residual cycle component, c_t.

When $\psi(L) = \theta(L)/\phi(L)$, Newbold and Vougas (1996) derived a computationally efficient algorithm for computing the components of (21). Alternatively, the components may be computed using the Kalman filter applied directly to the SSOE model (21). Oh and Zivot (2006) extended the method of Morley (2002) for cases in which $\Delta^2 y_t = \mathbf{z}'\boldsymbol{\alpha}_t$, where $\mathbf{z}$ is an $m \times 1$ vector with fixed elements and the $m \times 1$ state vector $\boldsymbol{\alpha}_t$ follows the transition equation (6). They showed that the components may be computed using

$$\begin{aligned} \tau_t^{BN} &= y_t - \mathbf{z}'\mathbf{T}^2(\mathbf{I}_m - \mathbf{T})^{-2}\boldsymbol{\alpha}_{t|t} \\ d_t^{BN} &= \Delta y_t + \mathbf{z}'\mathbf{T}(\mathbf{I}_m - \mathbf{T})^{-1}\boldsymbol{\alpha}_{t|t} \\ c_t^{BN} &= \mathbf{z}'\mathbf{T}^2(\mathbf{I}_m - \mathbf{T})^{-2}\boldsymbol{\alpha}_{t|t} \end{aligned} \tag{22}$$

4.2 The Relationship between the BN Decomposition and UC Models

Assume that $\psi(L) = \theta(L)/\phi(L)$ in (17), and consider (21) rewritten as a typical UC model with three shocks

$$\begin{aligned} y_t &= \tau_t + c_t \\ \tau_t &= \tau_{t-1} + d_{t-1} + w_t,\ w_t \sim \text{iid } (0, \sigma_w^2) \\ d_t &= d_{t-1} + u_t,\ u_t \sim \text{iid } (0, \sigma_u^2) \\ \phi(L)c_t &= \theta^*(L)\ v_t,\ v_t \sim \text{iid } (0, \sigma_v^2) \\ \text{cov}(w_t, u_t) &= \sigma_{wu},\ \text{cov}(w_t, v_t) = \sigma_{wv},\ \text{cov}(u_t, v_t) = \sigma_{uv} \end{aligned} \tag{23}$$

where the order of $\theta^*(L)$ is $n = \max(q-2, 0)$. The reduced form of (23) is

$$\phi(L)(1-L)^2 y_t = \phi(L)u_{t-1} + \phi(L)(1-L)w_t + (1-L)^2\theta^*(L)v_t \tag{24}$$

The MA polynomial has the order of $\max(p+1, n+2)$ with respect to L. From the MA portion, $\max(p+1, n+2)+1$ moments may be calculated. Excluding the AR parameters, the unknown parameters in the UC model (23) are the n MA parameters and the 6 covariance matrix parameters $\sigma_w^2, \sigma_u^2, \sigma_v^2, \sigma_{wu}, \sigma_{wv}$ and σ_{uv}.

If $(n+6) > (\max(p+1, n+2)+1)$ or $(q+2) > p+1$ when $q \geq 2$, the model (23) is under-identified. In this case, we can match the moments of the reduced form and the moments of the corresponding ARIMA model given different choices for the correlations. Admissible choices must satisfy certain necessary conditions such as positive definitiveness of the covariance matrix and invertibility of the MA coefficients.

4.3 ARIMA(0,2,2) Model

Connections between the BN decomposition, UC models, and some commonly used signal extraction filters can be illustrated using the following ARIMA(0,2,2) reduced form model for y_t

$$(1-L)^2 y_t = (1 + \theta_1 L + \theta_2 L^2)\epsilon_t \tag{25}$$

This is the unrestricted reduced form associated with the following UC-ARMA(0,0) model

$$\begin{aligned} y_t &= \tau_t + c_t \\ \tau_t &= \tau_{t-1} + d_t + w_t,\ w_t \sim \text{iid } (0, \sigma_w^2) \\ d_t &= d_{t-1} + u_t,\ u_t \sim \text{iid } (0, \sigma_u^2) \\ c_t &= v_t,\ v_t \sim \text{iid } (0, \sigma_v^2) \\ \text{cov}(w_t, u_t) &= \sigma_{wu},\ \text{cov}(w_t, v_t) = \sigma_{wv},\ \text{cov}(u_t, v_t) = \sigma_{uv} \end{aligned} \tag{26}$$

The reduced form of (26) is

$$(1-L)^2 y_t = u_{t-1} + (1-L)w_t + (1-L)^2 v_t \tag{27}$$

which implies an ARIMA(0,2,2) model. Not all of the parameters of (27) are identified since $q+2 = 4 > p+1 = 1$. To determine which parameters may be identified, consider the moments of (25)

$$\begin{aligned} \gamma_0 &= 2\sigma_w^2 + \sigma_u^2 + 6\sigma_v^2 + 2\sigma_{wu} + 6\sigma_{wv} + 2\sigma_{uv} \\ \gamma_1 &= -\sigma_w^2 - 4\sigma_v^2 - \sigma_{wu} - 4\sigma_{wv} - 2\sigma_{uv} \\ \gamma_2 &= \sigma_v^2 + \sigma_{wv} + \sigma_{uv} \end{aligned} \tag{28}$$

Notice that the variance of the slope shock is identified since it only depends on the reduced form moments, $\sigma_u^2 = \gamma_0 + 2\gamma_1 + 2\gamma_2$. However, the remaining parameters are not identified without further restrictions. All admissible UC models will satisfy the moment conditions (28), the invertibility of the MA polynomial in (25), the positive definiteness of the covariance matrix of $(w_t, v_t, u_t)'$, and will admit the same trend-cycle decomposition as the BN decomposition based on (25).

The SSOE model imposes three restrictions $|\rho_{wu}| = |\rho_{wv}| = |\rho_{uv}| = 1$, which exactly identifies the remaining parameters. Evaluating $\psi(1)$, $\tilde{\psi}(1)$ and $\tilde{\tilde{\psi}}(L)$ in (21) gives the resulting UC-ARMA(0,0) parameters in terms of the reduced form ARIMA(0,2,2) parameters

$$\begin{aligned}
\sigma_w^2 &= \xi_1^2\sigma_\epsilon^2,\ \sigma_u^2 = \xi_2^2\sigma_\epsilon^2,\ \sigma_v^2 = \xi_2^2\sigma_\epsilon^2 \\
\rho_{wv} &= \frac{\xi_1\xi_2}{|\xi_1\xi_2|},\ \rho_{wu} = \frac{\xi_1\xi_3}{|\xi_1\xi_3|},\ \rho_{uv} = \frac{\xi_2\xi_3}{|\xi_2\xi_2|} \\
\xi_1 &= \tilde{\psi}(1) + \psi(1) = 1 - \theta_2,\ \xi_2 = \psi(1) = 1 + \theta_1 + \theta_2,\ \xi_3 = \theta_2
\end{aligned}$$

The signs of the correlations depend on the signs of θ_1 and θ_2.

The traditional local linear trend model (e.g., Harvey, 1989) sets $\rho_{wu} = \rho_{wv} = \rho_{uv} = 0$. With these restrictions, the remaining variance parameters may be recovered using

$$\begin{bmatrix} \sigma_w^2 \\ \sigma_u^2 \\ \sigma_v^2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 6 \\ -1 & 0 & -4 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{bmatrix}$$

This model will be admissible provided all of the variances are positive. However, as discussed by Harvey (1989), the parameter space for the ARIMA(0,2,2) that supports the local linear trend model is quite restrictive. As a result, the filtered estimates of trend and cycle from the local linear trend model are likely to be different from those computed from the BN decomposition.

As shown by Harvey and Jaeger (1993) and Gomez (1999), the HP filter (e.g., Hodrick and Prescott, 1997) results from a restricted version of the local linear trend model. HP defined the permanent component as the solution to

$$\min_{\tau_1,\ldots,\tau_T} \sum_{t=1}^{T}(y_t - \tau_t)^2 + \lambda\sum_{t=3}^{T}((1-L)^2\tau_t)^2 \tag{29}$$

where λ is a smoothness parameter such that large values of λ produce smooth trends. For quarterly data HP recommended using $\lambda = 1600$. This solution is equivalent to the smoothed estimate of τ_t from the local linear trend model with the additional restrictions $\sigma_w^2 = 0$ and $\sigma_v^2/\sigma_u^2 = \lambda$. The resulting cycle from the HP detrended data is equivalent to the smoothed estimate of c_t from the restricted local linear trend model. If λ is fixed then there are no parameters to be estimated, and the HP filter imposes two overidentifying restrictions which may be tested against the unrestricted reduced form.

Gomez (2001) showed that certain Butterworth or bandpass filters have more desirable properties than the HP filter and also have UC model representations. For example, consider the Butterworth filters based on the sine and tangent functions as described in Gomez (2001) and denoted BFS and BFT, respectively. These filters depend on a differencing parameter d and a frequency parameter x_c. For the BFS, Gomez (2001) showed that $\lambda = [2\sin(x_c/2)]^{-2d}$ where λ is the smoothness parameter for the HP filter in (29). For example, $\lambda = 1600$ is equivalent to $x_c = 0.1583$, or a period of 9.2 years. Using the results in the Appendix of Gomez (2001), it can be shown that the parameters of the BFS and BFT when $d = 2$ can be mapped into the parameters of the ARIMA(0,2,2) reduced form. For the BFS, the procedure is as follows. Set

$\lambda = [2\sin(x_c/2)]^{-4}$, compute $C = \sin(x_c/2)^2$, $D = 1 - 2C\cos(\pi/2) + C^2$ and $E = \sqrt{(C+\sqrt{D})^2 - 1}$. Then the ARIMA(0,2,2) parameters are determined using

$$\theta_1 = \frac{2(C - \sqrt{D})}{C + \sqrt{D} + E},\ \theta_2 = \frac{C + \sqrt{D} - E}{C + \sqrt{D} + E},\ \sigma_\epsilon^2 = \lambda(C + \sqrt{D} + E)^2$$

For the BFT, set $\lambda = 1/\tan(x_c/2)^4$, compute $C = \tan(x_c/2)$, $D = \cos(\pi/2)$, and $E = C\sqrt{2(1-D)}$. Then the ARIMA(0,2,2) parameters are determined using

$$\theta_1 = \frac{2(C^2 - 1)}{C^2 + 1 + E},\ \theta_2 = \frac{C^2 + 1 - E}{C + 1 + E},\ \sigma_\epsilon^2 = \lambda(C^2 + 1 + E)^2$$

As a result, the appropriateness of these filters can be tested against the reduced form ARIMA(0,2,2) model.

5 Illustration Using US Real GDP

In this section we illustrate the relationship between the BN decomposition and various UC models using U.S. postwar quarterly real GDP data from 1947:I through 2007:I.[6] We first consider I(1) models and then I(2) models.

5.1 I(1) Models

Figure 4 shows the log quarterly growth rate in percent along with the first 10 sample autocorrelations. The first two autocorrelations are clearly non-zero and there appears to be a cyclical pattern in the higher order autocorrelations. Determining the most appropriate ARIMA(p,1,q) forecasting model for real GDP growth to compute the BN decomposition is a difficult task (e.g., see Campbell and Mankiw, 1987). Standard model selection criteria (e.g., AIC and BIC) tend to select low order ARIMA$(p,1,q)$ models and the resulting BN cycles tend to be noisy and lack business cycle features. We recommend using the ARIMA$(p,1,q)$ model that is the unrestricted reduced form associated with the most general UC model to be considered. This allows the reduced form model to potentially capture the dynamics implied by the UC models. In addition, any restrictions imposed by the UC models can be directly evaluated by comparing likelihood values. We consider the UC-ARIMA(2,1) model (13) as the most general model and use the unrestricted ARIMA(2,1,2) model to compute the benchmark BN decomposition. To support this choice, we also fit all ARIMA$(p,1,q)$ models with $p, q \leq 3$ and found that the ARIMA(2,1,2) is preferred by the AIC and the ARIMA(1,1,0) is preferred by the BIC.

Table 1 shows maximum likelihood (ML) estimates, under the assumption of normally distributed errors, of the ARIMA(2,1,2) model, of the identified UC-ARMA(2,1) models discussed in subsection 3.3 as well as the estimated moments γ_0, γ_1 and γ_2 derived from the MA portion of the estimated reduced form

[6] The data were obtained from the FRED database at the Federal Reserve Bank of St. Louis.

	ARIMA(2,1,2)		UC models			
	Estimate	(std err)	SSOE	MNZ	UC0	Proietti
μ	0.3453	(0.0718)	-	-	-	-
ϕ_1	1.3649	(0.1452)	1.3649	1.3649	1.4971	1.4851
ϕ_2	-0.7819	(0.1747)	-0.7819	-0.7819	-0.5687	-0.5636
θ_1	-1.1100	(0.2176)	-	-	-	-
θ_2	0.6225	(0.2268)	-	-	-	-
σ_ϵ	0.9049	(0.0413)	-	-	-	-
d	-	-	0.8279	0.8279	0.8309	0.8311
σ_ω	-	-	1.1118	1.1118	0.5998	0.6710
σ_υ			0.5486	0.5541	0.6293	0.5178
θ_υ			0.0646	0	0	0.2076
$\rho_{\omega\upsilon}$	-	-	-1	-0.9487	0	0
γ_0	2.1449	-	2.1449	2.1449		
γ_1	-1.4747	-	-1.4747	-1.4747		
γ_2	0.5097	-	0.5097	0.5097		
AR roots	$0.8727\pm0.7192i$					
MA roots	$0.8916\pm0.9008i$					
log-likelihood	-317.6529	-	-317.6529	-317.6529	-319.2908	-319.2572

Table 1: Estimates from the ARIMA(2,1,2) model and the corresponding UC models for U.S. real GDP.

ARIMA(2,1,2) model.[7] All of the estimated parameters in the ARIMA(2,1,2) are significantly different from zero, and the estimated persistence is $\hat{\psi}(1) = \hat{\theta}(1)/\hat{\phi}(1) = 1.229$. The SSOE and MNZ ($\rho_{wv}$ free and $\theta_v = 0$) models have the same likelihood value as the unrestricted reduced form and so are admissible models. In the SSOE model, $\rho_{wv} = -1$ (since $\hat{\psi}(1) > 0$) and $\hat{\theta}_v = -1.4789$ which implies a non-invertible model for the cycle. In the MNZ model, $\hat{\rho}_{wv} = -0.9487$. Although the model from Proietti (2006) (θ_v free and $\rho_{wv} = 0$) is just identified by the order condition, the lower likelihood value indicates that restrictions imposed by the model are not consistent with the unrestricted reduced form. The UC0 model ($\rho_{wv} = \theta_v = 0$) has the lowest likelihood and imposes one overidentifying restriction. The likelihood ratio statistic for testing the overidentifying restriction is 3.2758, with a p-value of 0.0703, which is moderate evidence against the restriction.

A useful diagnostic for evaluating fitted models of the business cycle involves comparing the moments of the actual data with the moments of simulated data from the fitted models. For example, Figure 4 also shows simulated data from the fitted ARIMA(2,1,2) model, the UC0 model, and the first ten population autocorrelations from the models[8]. The autocorrelations from the ARIMA(2,1,2) match those from the data whereas those from the UC0 model do not. This provides additional evidence against the UC0 model.

[7] All models were estimated using S-PLUS 7.0 with S+FinMetrics 2.0 as described in Zivot, Wang and Koopman (2004) and Zivot and Wang (2006). S+FinMetrics 2.0 utilizes the algorithms in SsfPack developed in Koopman, Shephard, and Doornik (1999).

[8] The population autocorrelations from the fitted UC0 model are estimated by averaging the sample autocorrelations from 500 simulated samples.

The full set of admissible UC-ARMA(2,1) models is illustrated in Figure 3. This set is constructed by finding all values of ρ_{wv}, θ_v and σ_v, with σ_w^2 fixed according to (16) and the AR parameters fixed at the ARIMA(2,1,2) values, such that the moment conditions (15) are satisfied. The figure shows that the permissible range of correlation values, ρ_{wv}, is between about -0.76 and -1. For these values of ρ_{wv}, the signal-to-noise ratio σ_w/σ_c varies from about 1.82 to 5.5. The set also shows that there is an invertible SSOE model with $\rho_{wv} = -1$ and $\theta_v \approx 0.06$.

The range of permissible values for ρ_{wv} depends on the assumption of a UC-ARMA(2,1) model. Assuming different dynamics for the cycle will generally imply a different range for ρ_{wv}. However, some general results can be established. The sign of ρ_{wv} is related to $\psi(1)^2$ computed from the reduced form ARIMA model. For example, Lippi and Reichlin (1992) showed that if $\rho_{wv} = 0$ then it must be the case that $\psi(1)^2 < 1$. Recently, Nakagura and Zivot (2006) showed that $\psi(1)^2 \geq 1$ implies $\rho_{wv} < 0$ and that $\rho_{wv} < -\sqrt{1 - V^{-1}}$ provided $V > 1$ where $V = \psi(1)^2\sigma_\epsilon^2/\mathrm{var}(\Delta y_t)$.

The cyclical component c_t^{BN} from the BN decomposition computed from the ARIMA(2,1,2) and the filtered cyclical estimate $c_{t|t}$ computed from the SSOE and MNZ models are reported in Figure 5. The BN decomposition is computed using (7) with

$$\boldsymbol{\alpha}_t = \begin{bmatrix} \Delta^2 y_t \\ \phi_2 \Delta^2 y_{t-1} + \theta_1\epsilon_t + \theta_2\epsilon_{t-1} + \theta_3\epsilon_{t-2} \\ \theta_2\epsilon_t + \theta_3\epsilon_{t-1} \\ \theta_3\epsilon_t \end{bmatrix}, \ \mathbf{T} = \begin{bmatrix} \phi_1 & 1 & 0 & 0 \\ \phi_2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and $\mathbf{z}' = (1, 0, 0, 0)'$. The BN cycle is identical to the filtered cycles except for the first observation. The key difference between the decompositions is that in the MNZ model the cycle is estimated with error. Accordingly, $c_{t|t}$ is plotted with $\pm 2 \times SE$ bands computed from the output of the Kalman filter which shows considerable uncertainty about the estimated real-time cycle.[9]

All of the UC models that match the moments of the data have the same filtered estimates, but they have different smoothed estimates. To illustrate this point, Figure 6 presents smoothed cycle estimates $c_{t|T}$ from UC models with $\rho_{wv} = -1$, -0.9, and -0.8 that are consistent with the estimated ARIMA(2,1,2) model. The smoothed estimates from the SSOE model are identical to the filtered estimates since the filtered estimates are computed with zero mean squared error. The smoothed estimates for models with $\rho = -0.9$ and $\rho = -0.8$ are substantially different from the filtered estimates and attribute much more variability to the cycle. Harvey and Koopman (2000) show that this behavior is due to the asymmetric nature of the weights in the smoothing algorithm from UC models. Negatively correlated shocks will put more weight on future observations than on past observations.

BN decomposition is often criticized because the reduced form ARIMA model is not well suited for cap-

[9] The confidence bands do not take into account sampling uncertainty associated with the estimated model parameters. Bayesian methods could be used to produce highest posterior density intervals for the filtered cycles that incorporate parameter uncertainty.

		ARIMA(2,1,2)		UC models	
	True value	Estimate	Implied UC Est	MNZ	UC0
μ	-	0.0411	-	-	-
ϕ_1	1.4971	1.6042	1.6042	1.6042	1.5960
ϕ_2	-0.5687	-0.6551	-0.6551	-0.6551	-0.6487
θ_1	-	-1.3623	-	-	-
θ_2	-	0.3990	-	-	-
σ_ϵ	-	0.9721	-	-	-
d	0.8309	-	0.8090	0.8090	0.8095
σ_ω	0.5998	-	0.7030	0.7030	0.7359
σ_v	0.6293	-	0.3151	0.4727	0.5266
θ_v	0	-	1.6235	0	0
$\rho_{\omega v}$	0	-	-1	0.2449	0
log-likelihood		-334.8434	-	-334.8434	-334.8602

Table 2: Estimates from the ARIMA(2,1,2) model and the corresponding UC models on simulated data from the UC0 model.

turing the subtle dynamics that may exist in the data. Often low order ARIMA models (e.g., ARIMA(0,1,1) or ARIMA(1,1,0)) are found to be the best fitting models by traditional model selection criteria, and these models produce simplistic cycles by construction. It is argued (e.g., Harvey and Jaeger, 1993) that structural UC models with orthogonal components can be tailored to capture business cycle dynamics better than reduced form ARIMA models. However, an appropriate reduced form ARIMA model can capture the same type of dynamic behavior as a structural UC-ARMA model. To illustrate this point, we simulated data from the fitted UC0 model given in column six of Table 1. We then fit the UC0, MNZ, ARIMA(2,1,2) models, and then computed the filtered cycles from each approach. The estimation results are given in Table 2, and the extracted cycles are compared in Figure 7. In terms of estimation, the AR parameters from all models are similar and the estimated correlation from the MNZ model is small and positive. The LR test for $\rho_{wv} = 0$ in the UC model is 0.0336 with a p-value of 0.8546. The top panel of Figure 7 shows that c_t^{BN} from the ARIMA(2,1,2) is very close to $c_{t|t}$ from the UC0 model. The bottom panel of Figure 7 compares c_t^{BN} from an underspecified ARIMA(1,1,0) to $c_{t|t}$ and illustrates the typical empirical result that the BN decomposition produces small noisy cycle estimates. Several points are worth noting here. First, the fact that the filtered cycle from the UC0 model is not roughly equal to the BN cycle when applied to U.S. real GDP indicates that either the restrictions are incorrect or that the UC0 model is misspecified. If the model was correct, they should be roughly the same. Secondly, imposing restrictions on the parameter space may not be helpful for a policymaker interested in real-time estimates. If the restrictions were valid, the BN decomposition would provide the cycle. Finally, contrary to popular belief, the BN decomposition can produce smooth cycles. It does not produce smooth cycles, however, for the U.S. real GDP data as illustrated by Figure 5.

	ARIMA(0,2,2)				UC models	
	Estimate	(std err)	HP-ARIMA	Single Source	Two Source	Two Source
θ_1	-0.7396	(0.0592)	-1.777	-	-	
θ_2	-0.2604	(0.0534)	0.7994	-	-	
σ_ϵ	0.9391	(0.0447)	44.7258	-	-	
σ_ω	-	-	-	1.1836	1.1836	0.9810
σ_u	-	-	-	0.0000	0.0000	0.0066
σ_υ	-	-	-	0.2445	0.5541	0.0000
$\rho_{\omega u}$	-	-	-	1	0	0
$\rho_{\omega\upsilon}$	-	-	-	-1	-0.9	0
$\rho_{u\upsilon}$	-	-	-	-1	0	0
γ_0	1.4241		9595.93	1.4241	1.4241	
γ_1	-0.4824		-6396.62	-0.4824	-1.4747	
γ_2	-0.2296		1599.16	-0.2296	-0.2296	
MA roots	$1.0000 + 0i$ $-3.8409 + 0i$		$1.111\pm$ $0.12467i$			
log-likelihood	-328.4879	-	-1132.94	-328.4879	-328.4879	-339.3109

Table 3: Estimates from the ARIMA(0,2,2) model and the corresponding UC models for U. S. real GDP.

5.2 I(2) Models

We focus initially on the ARIMA(0,2,2) model because it is the reduced form associated with the local linear trend model, the HP filter, and some of the BFS and BFT filters. Table 3 shows MLEs of the ARIMA(0,2,2) an identified UC-ARMA(0,0) model with $\rho_{wv} = -0.9$ and $\rho_{wu} = \rho_{uv} = 0$, the local linear trend model, and implied estimates for some other models. One of the MA roots in the ARIMA(0,2,2) is almost unity which indicates potential overdifferencing of the data. This is equivalently reflected by the near zero estimate of σ_u in the UC-ARMA(0,0) models. Figure 9 shows the combinations of ρ_{wv}, ρ_{wu} and ρ_{uv} in the UC-ARMA(0,0) model that produce the same likelihood value as the ARIMA(0,2,2) model. The figure shows that the correlation between the trend and cycle shocks, ρ_{wv}, cannot take values higher than about -0.8, whereas ρ_{wu} and ρ_{uv} appear not to be restricted. This result explains the lower likelihood value for the local linear trend model which imposes zero correlation between all pairs of shocks. Column four in Table 3 gives the ARIMA(0,2,2) model estimates implied by the HP filter restrictions on the local linear trend model. The restrictions implied by the HP filter are clearly rejected by the data. Figure 8 shows the BN cycle computed from (22) with

$$\boldsymbol{\alpha}_t = \begin{bmatrix} \Delta y_t \\ \epsilon_t \\ \epsilon_{t-1} \end{bmatrix}, \; \mathbf{T} = \begin{bmatrix} 0 & \theta_1 & \theta_2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

where $z = (1, 0, 0, 0)'$. The BN cycle is smaller in magnitude and noisier than the corresponding cycle from the ARIMA(2,1,2) model

A more realistic model is the ARIMA(2,2,3), which is the reduced form associated with the stochastic slope UC-ARMA(2,0) and UC-ARMA(2,1) models used by Clark(1987) and Harvey and Jaeger (1993), respectively. Table 4 reports the MLEs for the ARIMA(2,2,3). As with the ARIMA(0,2,2) model, there is a near unit moving average root in the ARIMA(2,2,3) model.

In the UC-ARMA(2,1) model, not all of the parameters are identified since $q+2=5>p+1=3$. The moment conditions from (24) used for identification are

$$\begin{aligned}
\gamma_0 &= 2(1+\phi_1+\phi_1^2-\phi_1\phi_2+\phi_2^2)\sigma_w^2+(1+\phi_1^2+\phi_2^2)\sigma_u^2+2(3-4\theta_v+3\theta_v^2)\sigma_v^2 \\
&\quad +2(1+\phi_1(\phi_1+1)-\phi_2(\phi_1-\phi_2))\sigma_{wu}+2(3+3\phi_1-\phi_2-\theta_v-3\phi_1\theta_v+3\phi_2\theta_v)\sigma_{wv} \\
&\quad +2(1-\phi_1(\theta_v-2)-\phi_2(1-2\theta_v))\sigma_{uv} \\
\gamma_1 &= (-(\phi_1+1)+(\phi_1-\phi_2)(\phi_2-\phi_1-1))\sigma_w^2+(-\phi_1+\phi_1\phi_2)\sigma_u^2+(7\theta_v-4\theta_v^2-4)\sigma_v^2 \\
&\quad +((\phi_1+1)(\phi_2-1)-\phi_1(1+\phi_1-\phi_2)-\phi_2^2)\sigma_{wu}+(2(\phi_1-\phi_2)(\theta_v-1)+(1-2\theta_v) \\
&\quad (\phi_2-\phi_1-1)+(\theta_v-\phi_1)-3)\sigma_{wv}+((\theta_v-2)(1-\phi_2)-2\phi_1(1-\theta_v)-\theta_v\phi_2)\sigma_{uv} \\
\gamma_2 &= (\phi_1-\phi_2-\phi_2(\phi_1+1))\sigma_w^2-\phi_2\sigma_u^2+(1-2\theta_v+\theta_v(\theta_v-2))\sigma_v^2 \\
&\quad +(\phi_1-2\phi_2-\phi_1\phi_2)\sigma_{wu}+(\phi_1-3\phi_2+1+\theta_v(\phi_2-3-\phi_1))\sigma_{wv} \\
&\quad +(-\phi_2+1-2\theta_v-\phi_1\theta_v)\sigma_{uv} \\
\gamma_3 &= \phi_2\sigma_w^2+\theta_v\sigma_v^2+\phi_2\sigma_{wu}+(\phi_2+\theta_v)\sigma_{wv}+\theta_v\sigma_{uv}
\end{aligned}$$

A simple derivation shows that the variance of the shock to the slope does not depend on any of the covariances.

$$\sigma_u^2=(\gamma_0+2(\gamma_1+\gamma_2+\gamma_3))/(1-2\phi_1-2\phi_2+2\phi_1\phi_2+\phi_1^2+\phi_2^2)$$

When the covariances $(\sigma_{wu},\sigma_{wv},\sigma_{uv})$ are given arbitrarily after identifying σ_u, the three unknowns $(\sigma_w,\sigma_v,\theta_v)$ are accordingly determined. Nonlinearity in the system of equations may result in multiple solutions. For example, fixing $(\rho_{wu},\rho_{wv},\rho_{uv})$ at $(1.0,-1.0,-1.0),(1.0,-0.95,-1.0)$ or $(1.0,-0.95,-0.95)$ in the UC-ARMA(2,1) model produces the same moments as the ARIMA(2,2,3) model and the resulting filtered estimates $c_{t|t}$ are the same as c_t^{BN}. The combinations of the correlations that are admissible are similar to those given in Figure 9 and are not reported. Whereas ρ_{wu} and ρ_{uv} are not bounded, ρ_{wv} is not allowed to take a value above -0.75.

In the UC-ARMA(2,0) model, the MA parameter cycle is set to zero so that one of the correlations may be estimated when the other two correlations are restricted. For example, if $\sigma_{wu}=\sigma_{uv}=0$ are imposed,

	ARIMA(2,2,3)		UC	
	Estimate	(standard error)	Single Source	Multi Sources
ϕ_1	1.3535	(0.1490)	1.3535	1.3535
ϕ_2	-0.7677	(0.1714)	-0.7677	-0.7677
θ_1	-2.0915	(0.2149)	-	-
θ_2	1.6985	(0.4208)	-	-
θ_3	-0.6069	(0.2144)	-	-
σ_ϵ	0.9069	(0.0415)	-	-
σ_w	-	-	1.1280	1.1281
σ_u	-	-	0.0001	0.0001
σ_v	-	-	0.2213	0.5836
θ_v	-	-	-1.4263	-
ρ_{wu}	-	-	1	0
ρ_{wv}	-	-	-1	-0.9455
ρ_{uv}	-	-	-1	0
AR roots	0.8815 ± 0.7249i			
MA roots	1.0001, 0.8993 ± 0.9159i			
Log likelihood	-318.4413		-318.4413	-318.4413

Table 4: Estimates from the ARIMA(2,2,3) model and the corresponding UC models for U. S. real GDP.

the remaining parameters $(\sigma_v^2, \sigma_w^2, \sigma_u^2, \sigma_{wv})'$ can be calculated from $\mathbf{\Phi}^{-1}\boldsymbol{\gamma}$, where

$$\mathbf{\Phi} = \begin{bmatrix} -6 & 2(1+\phi_1+\phi_1^2-\phi_1\phi_2+\phi_2^2) & 1+\phi_1^2+\phi_2^2 & 2(3+3\phi_1-\phi_2) \\ -4 & -1-2\phi_1+\phi_2+2\phi_1\phi_2-\phi_1^2-\phi_2^2 & -\phi_1+\phi_1\phi_2 & -4-4\phi_1+3\phi_2 \\ 1 & \phi_1-2\phi_2-\phi_1\phi_2 & -\phi_2 & 1+\phi_1-3\phi_2 \\ 0 & \phi_2 & 0 & \phi_2 \end{bmatrix}$$

$$\boldsymbol{\gamma} = \begin{bmatrix} \sigma_\epsilon^2(1+\theta_1^2+\theta_2^2+\theta_3^2) \\ \sigma_\epsilon^2(\theta_1+\theta_1\theta_2+\theta_2\theta_3) \\ \sigma_\epsilon^2(\theta_2+\theta_1\theta_3) \\ \sigma_\epsilon^2\theta_3 \end{bmatrix}$$

The MLEs of the stochastic slope UC-ARMA(2,0) reported in Table 4 show $\hat{\sigma}_u \approx 0$ and $\hat{\rho}_{wv} = -0.9455$ and are very close to the MLEs of the MNZ model reported in Table 1. The BN cycle and filtered cycles from the admissible UC-ARMA(2,1) models are essentially identical to those from the ARIMA(2,1,2) model.

6 Conclusion

The purpose of this paper was to clarify the relationship between the BN decomposition and UC models with correlated shocks and to understand what information the BN decomposition can provide. The BN decomposition does convey some information about the correlation between permanent and transitory shocks. The BN decomposition also imparts information on the fit of certain UC models and non-model based filters, whose trends are random walks.

Univariate models have a limited ability to construct good forecasting models to obtain the BN cycle. UC models and other non-model based filters try to provide additional information by forcing restrictions on the data. For policymakers interested in real-time estimates of the cycle, placing restrictions on the process may not be helpful. When the restrictions are valid and there actually is zero correlation between components, the BN decomposition has the ability to extract a smooth cycle and can provide similar filtered estimates. Correlation plays an important role in determining smoothed estimates and these smoothed estimates may not be identified. While Harvey and Koopman (2000) argue against UC models with correlated components because they may result in odd weighting patterns, the odd weighting patterns are a result of the data and may indicate misspecification of the model. The disagreement between the filtered UC estimates and the BN decomposition also provides further evidence along these lines.

Moving to multivariate forecasting models is one way to produce more realistic decompositions using the BN decomposition. Results in Evans and Reichlin (1994) indicate that better forecasting models will reduce the variability of the forecast error of the trend. This reduces the variability of the BN trend and consequently produces smoother cycles. We believe users should focus on developing better forecasting models using multivariate series that contain more information. Finally, while the BN decomposition has many uses, we think users should hesitate to interpret the SSOE model as structural. Morley (2006) recently discussed the economic and statistical implications of the SSOE representation and argued that it is not plausible for US real GDP.

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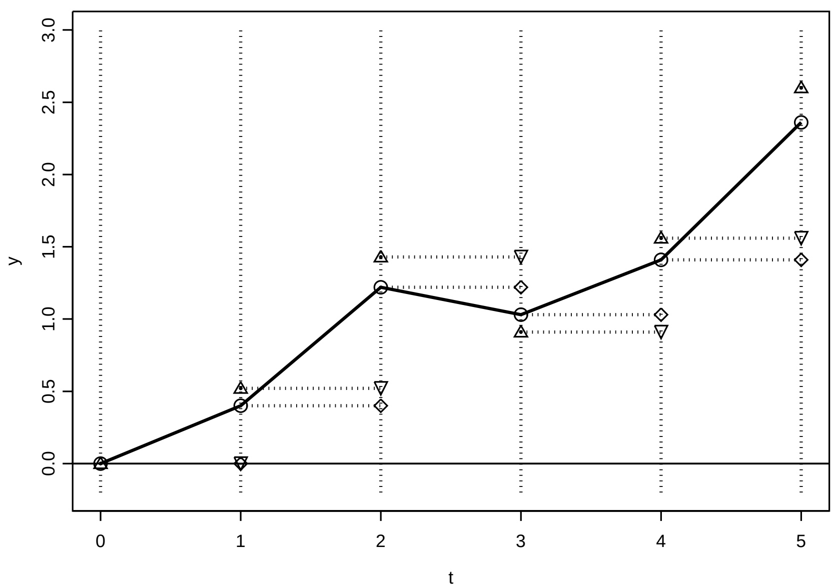


Figure 1: BN decomposition of an ARIMA(0,1,1) process with $\theta = 0.3$.

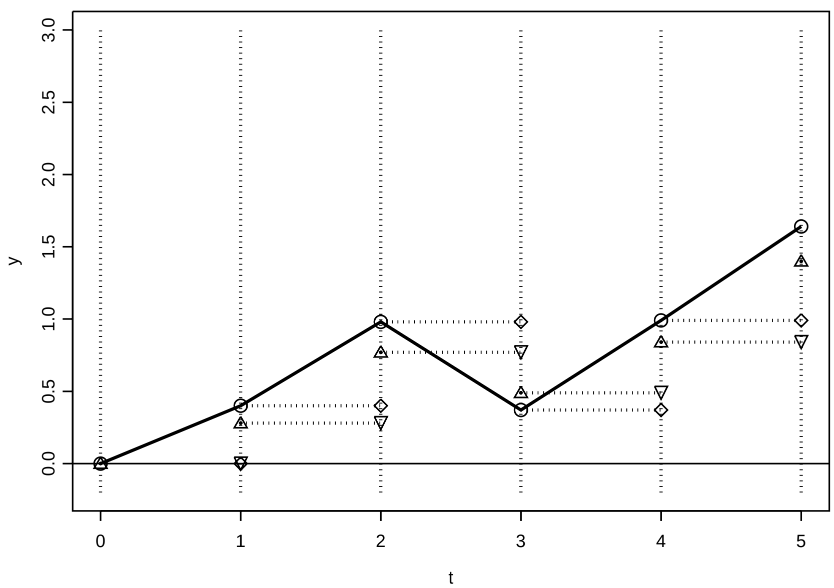


Figure 2: BN decomposition of an ARIMA(0,1,1) process with $\theta = -0.3$.

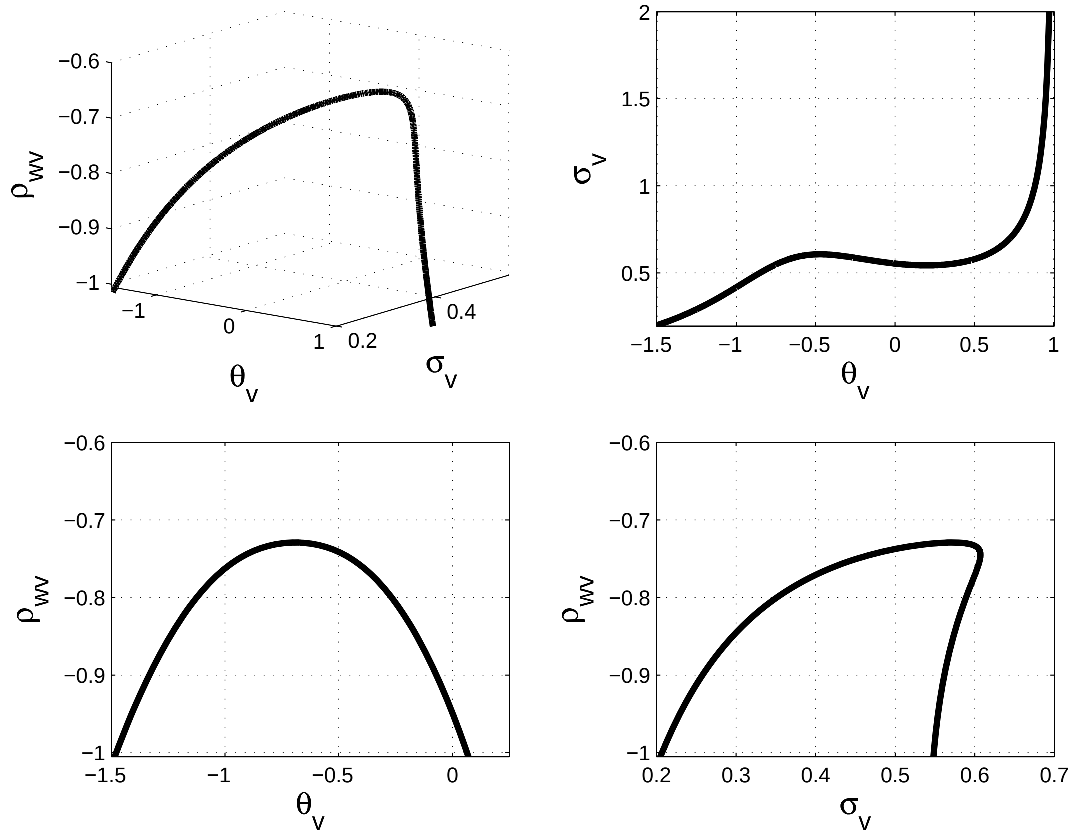


Figure 3: Combinations of θ_v, σ_v and ρ_{wv} that satisfy the moment equations (15) from an ARIMA(2,1,2) model for U.S. real GDP.

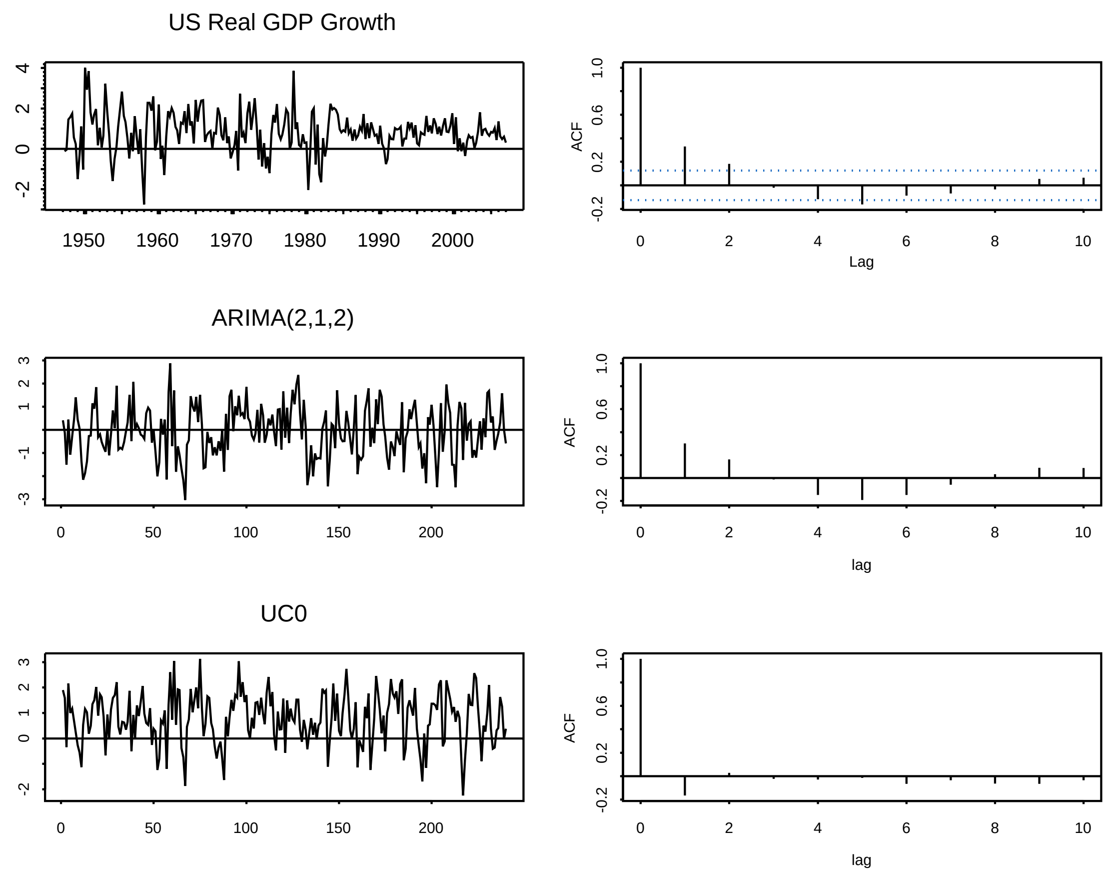


Figure 4: Top: U.S. real GDP growth and SACF. Middle: Simulated data from ARIMA(2,1,2) and ACF. Bottom: Simulated data from UC0 and ACF.

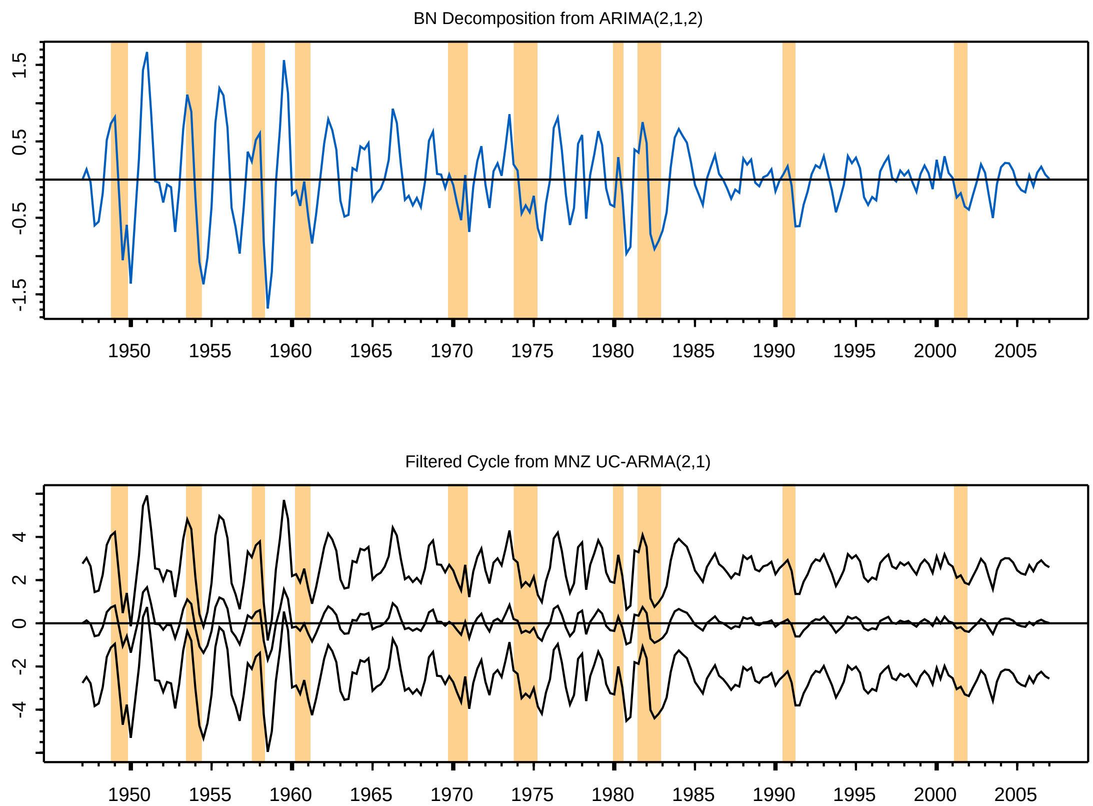


Figure 5: BN decomposition for U.S. real GDP. Top: c_t^{BN} from ARIMA(2,1,2) model and $c_{t|t}$ from SSOE model. Bottom: $c_{t|t}\pm$ 2 SE from MNZ model.

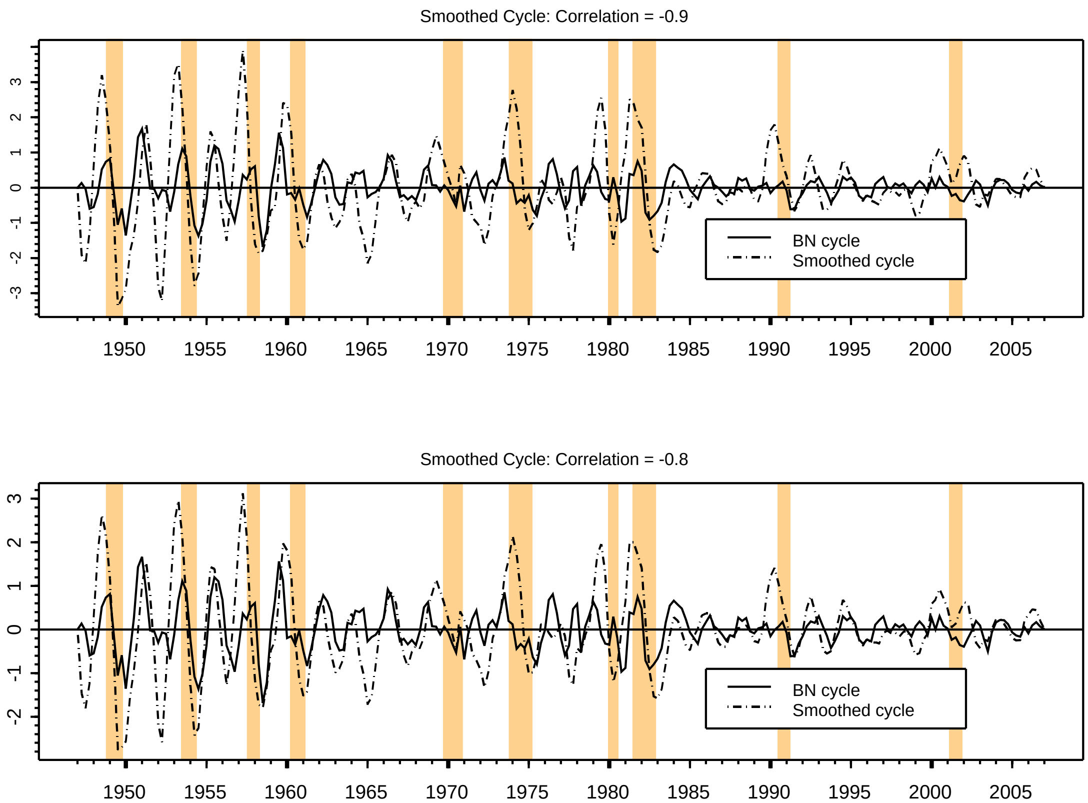
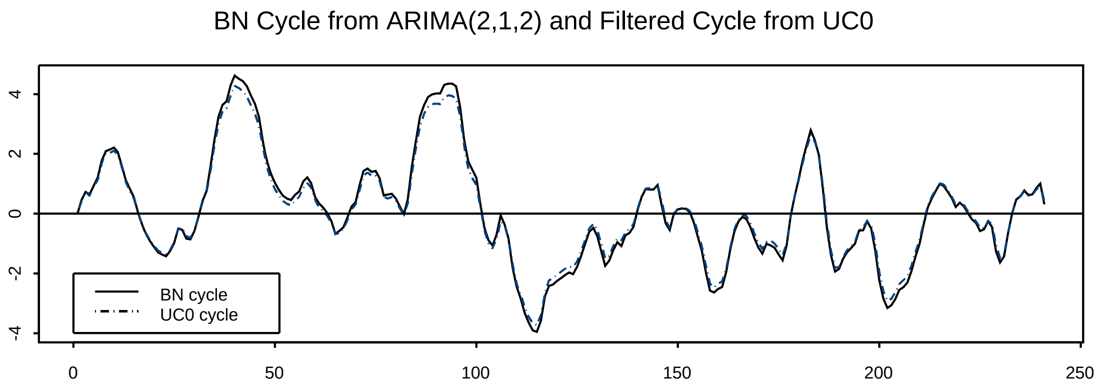


Figure 6: Smoothed cycles from UC-ARMA(2,1) models for U.S. real GDP.



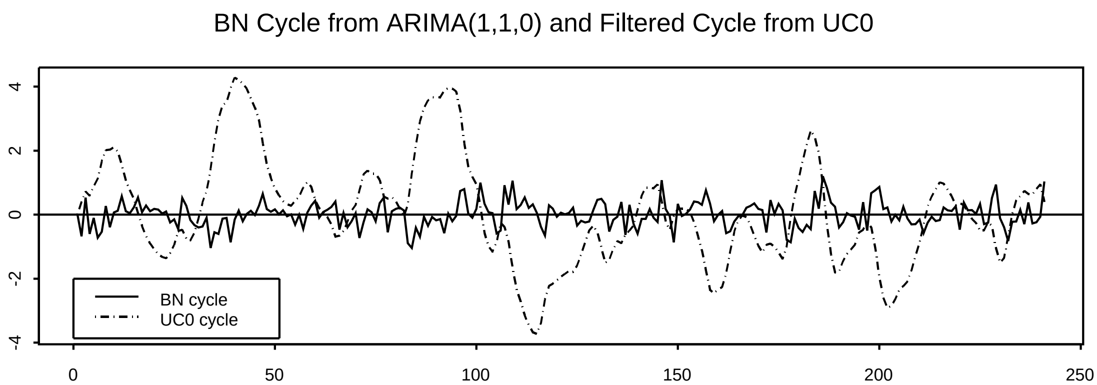


Figure 7: BN cycles, c_t^{BN}, and UC0 filtered cycles, $c_{t|t}$, from models fit to simulated data from the UC0 model.

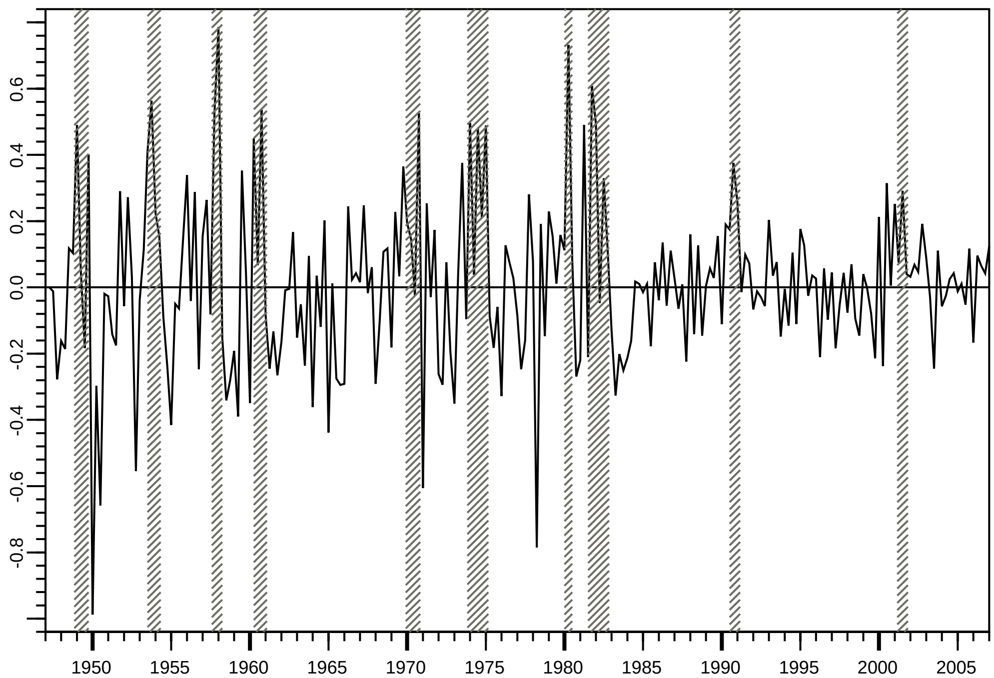


Figure 8: BN decomposition from the ARIMA(0,2,2) model for U.S. real GDP.

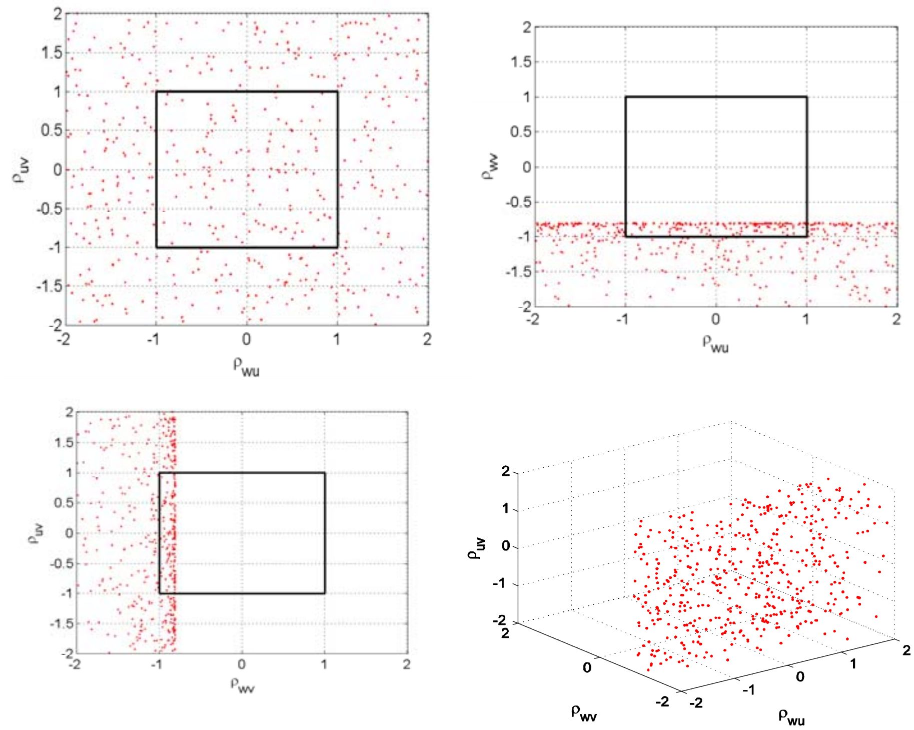


Figure 9: Combinations of ρ_{wv}, ρ_{wu} and ρ_{uv} that satisfy the moment equations (28) from an ARIMA(0,2,2) model for U.S. real GDP.