The Clark Model with Correlated Components

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Abstract

This paper is an extension of "Why are the Beveridge-Nelson and Unobserved-Components Decompositions of GDP so Different?" (Morley, Nelson, and Zivot, 2003) to Clark's double-drift unobserved components model. We show that the reduced-form of the double-drift model is an ARIMA(2,2,3) model, and we discuss various restrictions for identifying the parameters of the double-drift model with correlated components. When shocks to the smooth trend and cycle are allowed to be correlated but forced to be uncorrelated with shocks to the drift, the Kalman filter estimates of the trend and cycle are identical to the estimates from the Beveridge-Nelson decomposition from the ARIMA(2,2,3) model and are similar to the estimates from Morley, Nelson and Zivot. We also find that alternative identification schemes are not supported by the data.

1 Introduction

When the real GDP is assumed to follow an unit root process, the Beveridge-Nelson (BN) decomposition (Beveridge and Nelson, 1981) and the Kalman filter signal extrac-

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tion from an unobserved component (UC) model have been widely used to decompose the real GDP into random walk trend and stationary cycle components. Trend and cycle extraction from a UC model is more flexible than the BN decomposition because it explicitly takes the structure of the trend and various sources of shocks into consideration, whereas the BN decomposition relies on a reduced form parameterization of an autoregressive integrated moving average (ARIMA(p,1,q)) process. A drawback of UC models is that the trend and cycle components are often not identified without restrictions placed on the parameters of the models. Even though some authors insist that economic theory may give adequate restrictions to identify the UC model parameters, the identification problem is not easily resolved as the characteristics of the extracted trend and cycle components are often sensitive to the form of restrictions imposed. The usual restriction to identify the trend and cycle components of the UC model applied to the real GDP is to assume that the unobserved shocks to the trend and cycle are independent (see Watson 1986, Clark 1987, Harvey and Jaeger 1992). This identification scheme produces a smooth trend, similar to a linear trend, and a pronounced cycle with typical business cycle features. The implication of the UC model decomposition for business cycle analysis is that shocks to the transitory cycle are more important for explaining the business cycle than shocks to the trend. In contrast, the BN decomposition derived from an unrestricted reduced form ARIMA model gives a highly variable trend and a mitigated cycle which implies that shocks to the trend are more important for explaining the business cycle than shocks to the cycle. Recently, Morley, Nelson, and Zivot (2003) (hereafter MNZ) showed that for certain UC models the correlation between the trend and cycle shocks is an identified parameter which implies that the assumption of uncorrelated trend and cycle components is an overidentifying assumption. In addition, they showed that when the trend and cycle components are allowed to be correlated, the extracted trend and cycle components from the Kalman filter are identical to those derived from the BN decomposition. Finally, for U.S. real GDP, they found that the overidentifying restriction of uncorrelated components is rejected by the data.

The UC model for the level of log real GDP analyzed by MNZ was a simplified version of the UC model used by Clark (1987) in which the trend followed a pure random walk with constant drift, the cycle was a stationary AR(2) process, and the shocks to the trend were allowed to be correlated with the shocks to the cycle. They chose this model because it implies a reduced form ARIMA(2,1,2) model for the growth rate of real GDP, and there is a simple mapping from the reduced form ARIMA(2,1,2) parameters to the UC model parameters. While this model is convenient for analytic purposes, it is not the model used by Clark and others in empirical applications. Clark allowed the drift to the random walk component to also follow a pure random walk to capture smooth changes in trend. This "double-drift" trend specification is the most common trend specification for empirical analysis with UC models (see Harvey 1985, Harvey and Jaeger 1993, Stock and Watson 1998, and Mills 2003). In this paper, we extend the analysis of MNZ to Clark's double-drift UC model. This extension poses the following technical problems:

1. Clark's double-drift UC model has three sources of shocks (two shocks to trend, and one shock to cycle), whereas MNZ's UC model only has two sources of shocks (one trend and one cycle). As a result, with correlated trend and cycle components the parameters of Clark's double-drift UC model are not identified without further restrictions.

2. Clark's double-drift UC model implies an ARIMA(2,2,3) reduced form model. As a result, real GDP follows a second order integrated, $I(2)$, process. To compare the trend and cycle components extracted from the UC model with those from the

BN decomposition, the BN decomposition from an $I(2)$ process must be derived. After addressing the above technical problems, we find that the logic of MNZ can be applied to Clark's double-drift UC model with some modifications. In particular, we show that Clark's double-drift UC model with uncorrelated components is overidentified. We show that the overidentifying restriction has weak support from the data, and that the filtered and smoothed estimates of the cycle and trend in real GDP are sensitive to values of the correlations between the cycle and trend shocks. In the double-drift UC model in which shocks to the smooth trend and cycle are allowed to be correlated and shocks to the drift are uncorrelated with the other shocks, we show that the extracted trend and cycle components are very close to those obtained by MNZ. We further show that alternative specifications in which shocks to the drift are allowed to be correlated with shocks to the smooth trend, or shocks to the cycle, are not supported by the data.

Our estimation results of the double-drift UC model with correlated components on U.S. real GDP support the view of Perron and Wada (2005) that there have been important changes to the drift function. Perron and Wada favor a single trend break in which the trend decreases about the time of the Oil Crisis of 1973. Our results suggest a more complex change in trend in which the decrease in trend starts in 1965, flattens in the early 1980s, and changes direction in the early 1990s. Contrary to Perron and Wada, however, our results for U.S. real GDP allowing for changes in trend and correlated components are not qualitatively different from those of MNZ.

The paper continues as follows. In Section 2, we present Clark's double-drift model, show that it has an ARIMA(2,2,3) reduced form representation, and discuss identification conditions when the shocks to the unobserved components are allowed to be correlated. In Section 3, we derive the BN decomposition from a reduced form ARIMA(2,2,3) model, and show by simulation the equivalence between the Kalman filter estimates of the trend and cycle components from a double-drift UC model and the BN decomposition estimates. In Section 4, we estimate an unrestricted ARIMA(2,2,3) model and various double-drift UC models using the postwar US real GDP data from MNZ. We give concluding remarks and suggestions for future research in Section 5. Our techincal results are provided in the Appendices.

2 Clark's Double-Drift UC model

In this section, we describe Clark's double-drift UC model with correlated components and discuss identification conditions for the parameters of the model.

2.1 Structural Representation

Clark's double-drift UC model is slightly different from the UC model of MNZ. In particular, Clark distinguished between "smooth trend" and "irregular trend." His model has the form

$$(1) \qquad y_t = \tau_t + c_t$$

$$(2) \qquad \tau_t = \tau_{t-1} + d_{t-1} + w_t,\ w_t \sim \text{iid } N(0, \sigma_w^2)$$

$$(3) \qquad d_t = d_{t-1} + u_t,\ u_t \sim \text{iid } N(0, \sigma_u^2)$$

$$(4) \qquad \phi(L)c_t = v_t,\ v_t \sim \text{iid } N(0, \sigma_v^2)$$

where y_t denotes the log of real GDP, τ_t denotes the unobserved overall (smooth) trend, d_t denotes the unobserved random walk (irregular) drift, and c_t denotes the unobserved AR(2) stationary cycle with $\phi(L) = (1 - \phi_1 L - \phi_2 L^2)$. The UC model (1) - (4) is often called the "double-drift" model since the drift d_t to the random walk trend τ_t also follows a random walk. As a result, the double-drift model implies that $y_t \sim I(2)$.

The double-drift model with correlated components has the additional parameters $\text{cov}(w_t, u_t) = \sigma_{wu}$, $\text{cov}(w_t, v_t) = \sigma_{wv}$, and $\text{cov}(u_t, v_t) = \sigma_{uv}$. To achieve identification, Clark assumed that the error terms v_t, w_t and u_t are mutually uncorrelated. In what follows, the UC model (1) - (4) with uncorrelated components will be referred to as the "double-drift UC0 model".

The model used by MNZ is a restricted version of (1) - (4) with $\sigma_u^2 = 0$ so that $d_t = d$. In addition, they allowed $\text{cov}(w_t, v_t) = \sigma_{wv} \neq 0$ and showed that the resulting model is exactly identified. Following MNZ, we will refer to this model as the "UCUR model", and we will refer to this model with the restriction $\sigma_{wv} = 0$ as the "UC0 model".

Using the same data as in MNZ[1], the maximum likelihood (ML) estimates for the parameters of the UC0 and double-drift UC0 models are given in Table 1. Our parameter estimates for the UC0 model match those of MNZ, and our estimates of the double-drift UC0 model parameters[2] are very similar to the UC0 model estimates. Even though the estimate of $\hat{\sigma}_u$ is small, its value is not exactly zero and avoids the so-called "pile-up" problem (see Stock and Watson 1998, and DeJong and Whiteman 1991) associated with moving average models with roots near the unit circle.

Since $\hat{\sigma}_u \neq 0$, the estimate of the drift d_t from the double-drift UC0 model is time varying. The filtered and smoothed estimates of the drift component ($\hat{d}_{t|t}$ and $\hat{d}_{t|T}$) are depicted in Figure 1. Both estimates indicate a generally decreasing trend function until the mid 1990s, with the filtered estimates being more volatile than the smoothed

[1]The quarterly real GDP data covers the period 1947:I to 1998:II. The UC models in the paper were estimated using S-PLUS 7.0 with S+FinMetrics 2.0 as described in Zivot, Wang, and Koopman (2004), and Zivot and Wang (2005). For comparison purposes, all the log likelihood values presented in the paper are computed by summing up the same number (204) of values from the prediction error decomposition of the log-likelihood.

[2]Clark(1987) reported 1.53, -0.59, 0.64, 0.01, and 0.74 as parameter estimates.

estimates, particularly at the beginning of the sample. The smoothed estimates show a slight decline through the late 1950s, level out and increase slightly until 1964, then drop steadily until the early 1990s at which they level off and begin to increase. Figure 2 contrasts the filtered and smoothed cycle estimates ($\hat{c}_{t|t}$ and $\hat{c}_{t|T}$) from the UC0 and double-drift UC0 models. The estimates from the two models have the same general shape. However, the filtered cycle estimates from the UC0 model lie above the double-drift UC0 estimates prior to 1980 and lie below afterwards. Moreover, the UC0 estimates are below trend throughout the 1990s whereas the double-drift UC0 estimates are above trend after 1996. The smoothed estimates from the two models agree much more closely than the filtered estimates. The main difference is that the UC0 estimates stay below trend through the 1990s whereas the double-drift estimates rise above trend after 1996.

Stock and Watson (1998) emphasized that when $\hat{\sigma}_u$ is close to zero, the usual asymptotic results for the ML estimator may not be reliable. In particular, $\hat{\sigma}_u$ can be biased toward zero and its distribution can be non-normal. For more reliable estimation and inference when σ_u is small, they proposed an alternative asymptotic theory based on the local-to-zero assumption

$$(5) \quad \sigma_u = \lambda/T$$

and developed asymptotically median unbiased estimates for λ and asymptotically valid confidence intervals for λ. The double-drift UC0 model (1)-(4) may be put in Stock and Watson's local-to-zero framework by rewriting it as follows:

$$(6) \quad \begin{aligned} \Delta y_t &= d_{t-1} + \zeta_t, \quad \zeta_t = w_t + \frac{\Delta v_t}{\phi(L)} \\ \Delta d_t &= (\lambda/T)\eta_t, \quad (\lambda/T)\eta_t = u_t \end{aligned}$$

where ζ_t is an ARMA(2,2) process. Under the assumption of uncorrelated shocks, the model satisfies the assumptions of Stock and Watson (1998). Using Nyblom's L statistic for testing the null hypothesis that $\lambda = 0$ (see Nyblom 1989), the median unbiased estimate of σ_u is 0.036, which is almost twice as large as the ML estimate reported in table 1, and the 90 percent confidence interval for σ_u is [0, 0.1662]. Figure 3 and 4 show the re-estimated trends and cycles from the double-drift UC0 model imposing the restriction that σ_u equals the median unbiased estimate ($\sigma_u = 0.036$). These estimates are very similar to the unrestricted ML estimates. The main difference is that the smoothed drift estimates show slightly larger variation.

2.2 Reduced Form ARIMA(2,2,3) Model and Identification of Structural UC Model

By rearranging equations (1)-(4), we get the following reduced form representation for the double-drift UC model:

$$(7) \quad \phi(L)(1-L)^2 y_t = (1-L)^2 v_t + \phi(L)(1-L)w_t + \phi(L)u_t$$

Since the MA terms on the right hand side of (7) indicate a maximum lag-length of 3, from the results on the aggregation of ARMA processes (see Hamilton 1994, pp 102 - 108), the model (7) can be equivalently described by the following ARIMA(2,2,3) process:

$$(8) \quad \phi(L)(1-L)^2 y_t = \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} + \theta_3\epsilon_{t-3}$$

We note that the structural UC model (1)-(4) may impose complicated restrictions on the parameters of (8), and that the autocorrelation structure of (8) with unrestricted parameters may not be compatable with the structural UC model.

With the AR polynomials on the left hand side of equations (7) and (8) being the same, we can derive a relationship between the parameters of the structural UC model and the reduced form ARIMA(2,2,3) model as follows. From the right-hand side of (7), the autocovariances of $\phi(L)(1-L)^2 y_t$ in terms of the structural UC model parameters are

$$
\begin{aligned}
(9) \quad \gamma_0 &= 6\sigma_v^2 + 2(1+\phi_1+\phi_1^2-\phi_1\phi_2+\phi_2^2)\sigma_w^2 + (1+\phi_1^2+\phi_2^2)\sigma_u^2 \\
&\quad + 2(3+3\phi_1-\phi_2)\sigma_{wv} + 2(1+2\phi_1-\phi_2)\sigma_{uv} \\
&\quad + 2(1+\phi_1+\phi_1^2-\phi_1\phi_2+\phi_2^2)\sigma_{wu} \\
\gamma_1 &= -4\sigma_v^2 + (-1-2\phi_1+\phi_2+2\phi_1\phi_2-\phi_1^2-\phi_2^2)\sigma_w^2 + (-\phi_1+\phi_1\phi_2)\sigma_u^2 \\
&\quad + (-4-4\phi_1+3\phi_2)\sigma_{wv} + 2(-1-\phi_1+\phi_2)\sigma_{uv} \\
&\quad + (-1-2\phi_1+\phi_2+2\phi_1\phi_2-\phi_1^2-\phi_2^2)\sigma_{wu} \\
\gamma_2 &= \sigma_v^2 + (\phi_1-2\phi_2-\phi_1\phi_2)\sigma_w^2 - \phi_2\sigma_u^2 + (1+\phi_1-3\phi_2)\sigma_{wv} \\
&\quad + (1-\phi_2)\sigma_{uv} + (\phi_1-2\phi_2-\phi_1\phi_2)\sigma_{wu} \\
\gamma_3 &= \phi_2(\sigma_w^2+\sigma_{wv}+\sigma_{wu}) \\
\gamma_j &= 0, \quad j \geq 4
\end{aligned}
$$

Similarly, from the right-hand side of (8), the autocovariances of $\phi(L)(1-L)^2 y_t$ in terms of the reduced form ARIMA(2,2,3) parameters are

$$
\begin{aligned}
(10) \quad \gamma_0 &= \sigma_\epsilon^2(1+\theta_1^2+\theta_2^2+\theta_3^2) \\
\gamma_1 &= \sigma_\epsilon^2(\theta_1+\theta_1\theta_2+\theta_2\theta_3) \\
\gamma_2 &= \sigma_\epsilon^2(\theta_2+\theta_1\theta_3) \\
\gamma_3 &= \sigma_\epsilon^2\theta_3 \\
\gamma_j &= 0, \quad j \geq 4
\end{aligned}
$$

From (9) and (10), there are only four non-zero autocovariance equations to determine the six unknown structural UC model parameters $(\sigma_w^2, \sigma_u^2, \sigma_v^2, \sigma_{wu}, \sigma_{wv}, \sigma_{uv})$. To satisfy the order condition for identification, at least two additional restriction are needed.

This result implies that the double-drift UC0 model, which imposes three restrictions ($\sigma_{wv} = \sigma_{wu} = \sigma_{uv} = 0$), is overidentified. Consequently, as in MNZ, a specification test of the double-drift UC0 model may be constructed by testing the overidentification restriction.

To solve the identification problem, we consider two options:

(i) Increase the number of autocovariance equations by increasing the lag order of the AR cycle.

(ii) Restrict two parameters of the structural UC model.

To satisfy the order condition for identification using option (i), we must extend the AR lag order for the transitory cycle to at least 4. The reduced form representation for the UC model then becomes an ARIMA(4,2,5):

$$\phi(L)(1-L)^2 y_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \theta_3 \epsilon_{t-3} + \theta_4 \epsilon_{t-4} + \theta_5 \epsilon_{t-5} \tag{11}$$
$$\text{where } \phi(L) = (1 - \phi_1 L - \phi_2 L^2 - \phi_3 L^3 - \phi_4 L^4)$$

The unrestricted ARIMA(4,2,5) model for real GDP, however, is likely to be an overparameterized model that is difficult to estimate precisely due to the presence of canceling or near canceling roots of the AR and MA polynomials. For example, Table 2 gives the ML estimates of the parameters in (11) for log real GDP. None of the parameters are estimated precisely, and one of the estimated MA roots lies on the unit circle.

In what follows, we will follow option (ii) and consider solving the identification problem by restricting some parameters of the double-drift UC model. We emphasize that the restrictions are completely arbitrary from the viewpoint of identification. However, from an economic viewpoint, we may prefer certain kinds of restrictions over others. For example, one may argue that it is economically sensible to restrict the correlation between the drift shock u_t and cycle shock v_t to be zero because the former is permanent and the latter is transitory. This type of restriction is commonly used in the UC model literature as well as the structural vector autoregression literature (see Watson, 1994). Given that there are three sources of shocks, we consider identification under the following sets of restrictions:

Case I: $\sigma_{wu} = \sigma_{uv} = 0 \quad (\sigma_{wv} \neq 0)$

Case II: $\sigma_{wu} = \sigma_{wv} = 0 \quad (\sigma_{uv} \neq 0)$

Case III: $\sigma_{wv} = \sigma_{uv} = 0 \quad (\sigma_{wu} \neq 0)$

In case I, the shocks to the drift, u_t, are uncorrelated with the shocks to the smooth trend, w_t, and shocks to the cycle, v_t. Since we allow for correlation between shocks to τ_t and C_t, the model is similar to the correlated components model of MNZ. Imposing the case I restrictions in (9) gives the system of equations

$$(12) \quad \boldsymbol{\gamma} = \boldsymbol{\Phi}_1 \boldsymbol{\sigma}_1$$

where $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \gamma_2, \gamma_3)'$ is the vector of reduced form autocovariances from (10), $\boldsymbol{\sigma}_1 = (\sigma_v^2, \sigma_w^2, \sigma_u^2, \sigma_{wv})'$, and

$$\boldsymbol{\Phi}_1 = \begin{bmatrix} -6 & 2(1+\phi_1+\phi_1^2-\phi_1\phi_2+\phi_2^2) & 1+\phi_1^2+\phi_2^2 & 2(3+3\phi_1-\phi_2) \\ -4 & -1-2\phi_1+\phi_2+2\phi_1\phi_2-\phi_1^2-\phi_2^2 & -\phi_1+\phi_1\phi_2 & -4-4\phi_1+3\phi_2 \\ 1 & \phi_1-2\phi_2-\phi_1\phi_2 & -\phi_2 & 1+\phi_1-3\phi_2 \\ 0 & \phi_2 & 0 & \phi_2 \end{bmatrix}$$

Provided $\boldsymbol{\Phi}_1$ has full rank, the UC model parameters may be recovered from the reduced form autocovariances from the relation $\boldsymbol{\sigma}_1 = \boldsymbol{\Phi}_1^{-1}\boldsymbol{\gamma}$. This mapping, however, does not by itself guarantee that the resulting structural error covariance matrix is positive definite. That is, the autocovariance structure from the unrestricted reduced form

ARIMA(2,2,3) model may not be compatable with the autocorrelation structure from the identified structural UC model.

In case II, we assume that shocks to the smooth trend, w_t, are uncorrelated with the remaining shocks but we allow for correlation between the shocks to d_t and C_t. Imposing these restrictions on (9) gives the system of equations

$$(13) \quad \boldsymbol{\gamma} = \boldsymbol{\Phi}_2 \boldsymbol{\sigma}_2$$

where $\boldsymbol{\sigma}_2 = (\sigma_v^2, \sigma_w^2, \sigma_u^2, \sigma_{uv})'$ and

$$\boldsymbol{\Phi}_2 = \begin{bmatrix} -6 & 2(1+\phi_1+\phi_1^2-\phi_1\phi_2+\phi_2^2) & 1+\phi_1^2+\phi_2^2 & 2(1+2\phi_1-\phi_2) \\ -4 & -1-2\phi_1+\phi_2+2\phi_1\phi_2-\phi_1^2-\phi_2^2 & -\phi_1+\phi_1\phi_2 & 2(-1-\phi_1+\phi_2) \\ 1 & \phi_1-2\phi_2-\phi_1\phi_2 & -\phi_2 & 1-\phi_2 \\ 0 & \phi_2 & 0 & 0 \end{bmatrix}$$

Provided $\boldsymbol{\Phi}_2$ has full rank, the UC model parameters may be recovered from the reduced form autocovariances from the relation $\boldsymbol{\sigma}_2 = \boldsymbol{\Phi}_2^{-1}\boldsymbol{\gamma}$. However, as in case I, the value of $\boldsymbol{\sigma}_2$ may not imply a positive definite structural covariance matrix.

In case III, the shocks to the cycle are uncorrelated with the shocks to the trends, and the shocks to trends are allowed to be correlated, which produces the system of equations

$$\boldsymbol{\gamma} = \boldsymbol{\Phi}_3 \boldsymbol{\sigma}_3$$

where $\boldsymbol{\sigma}_3 = (\sigma_v^2, \sigma_w^2, \sigma_u^2, \sigma_{wu})'$ and

$$\boldsymbol{\Phi}_3 = \begin{bmatrix} -6 & 2(1+\phi_1+\phi_1^2-\phi_1\phi_2+\phi_2^2) & 1+\phi_1^2+\phi_2^2 & 2(1+\phi_1+\phi_1^2-\phi_1\phi_2+\phi_2^2) \\ -4 & -1-2\phi_1+\phi_2+2\phi_1\phi_2-\phi_1^2-\phi_2^2 & -\phi_1+\phi_1\phi_2 & -1-2\phi_1+\phi_2+2\phi_1\phi_2-\phi_1^2-\phi_2^2 \\ 1 & \phi_1-2\phi_2-\phi_1\phi_2 & -\phi_2 & \phi_1-2\phi_2-\phi_1\phi_2 \\ 0 & \phi_2 & 0 & \phi_2 \end{bmatrix}$$

The matrix $\mathbf{\Phi}_3$, however, is singular since the second column is identical to the fourth column. Since the rank condition for identification is not satisfied, the parameters of the double-drift UC model in case III are not identified.

3 BN Decomposition from ARIMA(2,2,3) Model

To compare the estimated overall trend and cycle from the double-drift UC model with those from the BN decomposition, we need to derive the BN decomposition algorithm for an ARIMA(2,2,3) model. Our derivation of the BN decomposition is an application of Newbold and Vougas (1996) and Morley (2002) to an ARIMA(2,2,3) process. By definition, the BN trend is the long-run forecast of the level of the series minus any deterministic portion of the forecast. The BN cycle is then the gap between the current level of the series and the BN trend (see Figure 5). Formally, we have

$$
\begin{aligned}
(14) \quad BN_t &= \lim_{J\to\infty} E_t[y_{t+J} - DT_J] \\
&= y_t + \lim_{J\to\infty} E_t[J\Delta y_t + J\Delta^2 y_{t+1} + (J-1)\Delta^2 y_{t+2} + \cdots + \Delta^2 y_{t+J} - DT_J]
\end{aligned}
$$

where BN_t is the BN trend, DT_J is the deterministic trend, and $E_t[\cdot]$ denotes expectation conditional on information available at time t. The second line of (14) makes use of the fact that y_t is an $I(2)$ process. As shown in Newbold and Vougas, the BN decomposition of an $I(2)$ process produces not only estimates of trends and cycles but also irregular trends. Using the arguments of MNZ, these BN decomposition estimates will correspond to the Kalman filter estimates from a double-drift UC model that produces the same autocorrelation structure as the underlying ARIMA(2,2,3) model. Explicit representations for BN_t, DT_t^{BN}, and C_t^{BN} that are valid for an ARIMA(2,2,3) model are derived in Appendix I. There we show that each component has the form:

$$(15) \quad BN_t = y_t - \mathbf{Z}\mathbf{T}^2(\mathbf{I}_4 - \mathbf{T})^{-2}\boldsymbol{\alpha}_{t|t}$$

(16) $DT_t^{BN} = \Delta y_t + \mathbf{Z}\mathbf{T}(\mathbf{I}_4 - \mathbf{T})^{-1}\boldsymbol{\alpha}_{t|t}$

(17) $C_t^{BN} = \mathbf{Z}\mathbf{T}^2(\mathbf{I}_4 - \mathbf{T})^{-2}\boldsymbol{\alpha}_{t|t}$

where $\mathbf{Z} = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$, and

$$\boldsymbol{\alpha}_t = \begin{bmatrix} \Delta^2 y_t \\ \phi_2 \Delta^2 y_{t-1} + \theta_1 \epsilon_t + \theta_2 \epsilon_{t-1} + \theta_3 \epsilon_{t-2} \\ \theta_2 \epsilon_t + \theta_3 \epsilon_{t-1} \\ \theta_3 \epsilon_t \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} \phi_1 & 1 & 0 & 0 \\ \phi_2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

To illustrate the equivalence between the estimated BN trend and cycle from an unrestricted ARIMA(2,2,3) model and the Kalman filtered estimates of trend and cycle from double-drift UC models under the case I and case II restrictions, we performed the following simulation experiment. First, we chose the following parameters for the ARIMA(2,2,3) model: $\phi_1 = 1.44$, $\phi_2 = -0.62$, $\theta_1 = -2.10$, $\theta_2 = 1.42$, $\theta_3 = -0.30$ and $\sigma_\epsilon = 0.98$.[3] Next, we calculated the corresponding parameters of the double-drift UC models under the case I and case II restrictions using (12) and (13), respectively. These values are given in Table 3. The parameters of these double-drift UC models satisfy the positive definite covariance condition. Then, we simulated 300 observations from the case I double-drift UC model. Finally, we computed the BN trend, drift, and cycle estimates from the fixed ARIMA(2,2,3) parameters using (15) - (17), and the Kalman filter trend, drift, and cycle estimates from the double-drift UC model parameters in Table 3.

Figure 6 shows that the estimated BN cycle using (17) is equivalent to the Kalman filter estimates of c_t from the double-drift UC models models under the case I and

[3] Under the given parameters, AR roots are $1.1613 \pm 0.51411i$ and MA roots are 2.3824 and $1.1755 \pm 0.13206i$. The modulus of the roots are 1.27, 2.38, and 1.1829, respectively.

II restrictions. Although not shown, the estimated overall trend and irregular drift from the BN decomposition using (15) and (16) are also identical to the Kalman filter estimates of τ_t and d_t, respectively. This simulation example illustrates a case in which the BN decomposition of an unrestricted ARIMA(2,2,3) model produces the same estimated cycles and trends as two exactly identified double-drift UC model under different identifying restrictions. However, we stress that the BN decomposition from an unrestricted ARIMA(2,2,3) model may not be compatible with a given double-drift UC model under the case I or case II restrictions because the mapping between the reduced form model parameters and the UC model parameters may not produce a structural error covariance matrix that is positive definite. Indeed, in the next section, we show that this is the case with U.S. real GDP.

4 Empirical Results for U.S. Real GDP

In this section, we report maximum likelihood (ML) estimates of the reduced form ARIMA(2,2,3) model as well as double-drift UC models under the case I and case II restrictions using the same postwar U.S. log real GDP data as MNZ.

4.1 ML Estimation of Reduced Form ARIMA(2,2,3) Model

Table 4 reports the exact ML estimates of the parameters in the ARIMA(2,2,3) model (8).[4] All the estimates are significant at the 5 percent level. However, one of the

[4]When the parameters are estimated, additional restrictions are necessary. In maximizing the log-likelihood function, we imposed stationarity constraints on the AR parameters and invertibility constraints on MA parameters. For specific transformation equations and the procedure, see Appendix II.

estimated MA roots is very close to the unit circle which suggests a potential pile-up problem. The period of the cycle implied by the estimated AR roots is about 2.43 years, which is much shorter than the cycle period, 7.62 years, estimated from Clark's double-drift UC0 model.

Figure 7 shows the estimated trends and cycle computed from the BN decomposition of the estimated ARIMA(2,2,3) model. The cycle displays a much smaller amplitude and shorter period than the filtered cycle estimate from Clark's double-drift UC0 model, and is very similar to the BN cycle reported in MNZ based on an estimated ARIMA(2,1,2) model.[5] Interestingly, the estimate of the BN drift (Figure 8) is almost identical to the filtered estimate of d_t from the double-drift UC0 model (Figure 1).

If we assume that the ARIMA(2,2,3) model has a unit MA root, then the model collapses to an ARIMA (2,1,2) model. Suppose the data is generated by the following ARIMA(2,1,2) model with an intercept term:

$$\phi(L)(1-L)y_t = \mu^* + \epsilon_t + \theta_1^* \epsilon_{t-1} + \theta_2^* \epsilon_{t-2} \tag{18}$$

Multiplying both sides of (18) by $(1-L)$ and rearranging gives the following ARIMA(2,2,3) model with a unit MA root:

$$\phi(L)(1-L)^2 y_t = \epsilon_t + (\theta_1^* - 1)\epsilon_{t-1} + (\theta_2^* - \theta_1^*)\epsilon_{t-2} - \theta_2^* \epsilon_{t-3} \tag{19}$$

Table 5 reports the exact ML estimates of the ARIMA(2,1,2) model, and the implied parameter values of the ARIMA(2,2,3) model with a unit MA root. The ARIMA(2,2,3) model parameters implied by the estimated ARIMA(2,1,2) model parameters are very close to the direct ML estimates of the ARIMA(2,2,3) parameters given in Table 4.[6]

[5] The average difference between cycles from both models is just 0.0288, which is 5.5 percentage of standard deviation of the estimated cycle. If we take the different number of observations and initial stage of Kalman filter into consideration, the difference is negligible.

[6] To investigate whether the result depends on the data set, we also used real GDP data up to

If the ARIMA(2,2,3) reduced form model has a unit moving average root, then the double-drift UC model collapses to a UC model with a constant drift random walk trend. The near unit root in the estimated ARIMA(2,2,3) model indicates that the variance to the drift shock in the double-drift UC model is close to zero.

4.2 ML Estimates of the Double-Drift UC Models

In this subsection, we estimate by ML the double-drift UC models for the log of real GDP under the case I and case II restrictions. To do this, we put the models in state space form and compute the prediction error decomposition of the log-likelihood using the Kalman filter.[7] The measurement and transition equations for the UC models have the form

$$y_t = \mathbf{Z}\ \boldsymbol{\alpha}_t$$

$$\boldsymbol{\alpha}_t = \mathbf{T}\ \boldsymbol{\alpha}_{t-1} + \mathbf{R}\ \boldsymbol{\eta}_t,\ \ \boldsymbol{\eta}_t \sim \text{iid } N(\mathbf{0}, \mathbf{Q})$$

where $\mathbf{Z} = \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}$, $\boldsymbol{\alpha}_t = \begin{pmatrix} \tau_t & d_t & C_t & C_t \end{pmatrix}'$, $\boldsymbol{\eta}_t = \begin{pmatrix} w_t & u_t & v_t \end{pmatrix}'$, and

$$\mathbf{T} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \phi_1 & \phi_2 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ \mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ \mathbf{Q} = \begin{bmatrix} \sigma_w^2 & \sigma_{wu} & \sigma_{wv} \\ \sigma_{wu} & \sigma_u^2 & \sigma_{uv} \\ \sigma_{wv} & \sigma_{uv} & \sigma_v^2 \end{bmatrix}$$

2005:II from the Federal Reserve Bank of St Louis. Even though we cannot get convergence in the maximization, the result seems to show stronger evidence for having unit MA root in ARIMA(2,2,3) model.

[7]In the estimation, stationarity conditions on the autoregressive parameters and a positive definiteness condition on the innovation covariance matrix are imposed. The details of these restrictions are given in the Appendix II.

For the specification of the initial state vector, we utilize the exact initialization scheme of Koopman (1997) and described in Koopman, Shephard and Doornik (2001).

4.2.1 ML Estimation of Double-Drift Model under Case I

Table 6 gives the ML estimates of the UC model parameters under the case I restrictions $\sigma_{wu} = \sigma_{uv} = 0 \ \ (\sigma_{wv} \neq 0)$. All estimates, except $\hat{\sigma}_u$, are statistically different from zero at the 5 percent level. The estimated correlation between the smooth trend and the cycle is $\hat{\rho}_{wv} = -0.908$, which is very close to the corresponding estimate reported by MNZ. Table 6 also reports estimates of the UC model parameters implied by the estimated ARIMA(2,2,3) parameters using (12). The two sets of estimates are the same to the fourth decimal place, and the log-likelihood for the UC model is the same as the log-likelihood from the ARIMA(2,2,3) model. Using the log-likelihood from the double-drift UC0 model in Table 1 together with the case I log-likelihood from Table 6, we may compute a likelihood ratio (LR) test statistic for the overidentification restriction $\sigma_{wv} = 0$. This statistic is 2.0484, with a $\chi^2(1)$ p-value of 0.1523, which shows moderate evidence against the restriction $\sigma_{wv} = 0$.

Figures 8 and 9 show the filtered and smoothed estimates of the overall trend, irregular drift, and cycle from the double-drift model with $\sigma_{wu} = \sigma_{uv} = 0$. The filtered estimates of the cycle and trend components are almost identical to the BN decomposition estimates computed from the ARIMA(2,2,3) model. The filtered estimates of the irregular drift are very similar those from from Clark's double-drift UC0 model, even though the filtered cycle estimates are very different. This result occurs because both models assume that the drift shock is independent of the other shocks. The smoothed drift estimates have the same general shape as those from Clark's double-drift UC0 model but show less variation. As noted by Proietti (2003), the smoothed cycle estimates are much more variable than the filtered estimates and have subtantially different characteristics.

To investigate the robustness of the estimation, we re-estimated the model imposing the restriction $\sigma_u = 0.036$, which is the median unbiased estimate of σ_u from the double-drift UC0 model.[8] The resulting parameter estimates (Table 7), and trend, drift, and cycle estimates (Figure 10 and 11) show only minor differences.

4.2.2 ML Estimation of Double-Drift Model under Case II

Table 8 reports ML estimates for the double-drift model, and the implied estimates from the reduced form ARIMA(2,2,3) model using (13), under the case II restrictions $\sigma_{wu} = \sigma_{wv} = 0$ ($\sigma_{uv} \neq 0$). The ML estimates and log-likelihood value are similar to those from Clark's double-drift UC0 model reported in Table 1. In particular, the estimate of σ_{uv} is close to zero and has a moderately large estimated standard error which suggests that u_t and v_t are uncorrelated. However, this is misleading since the estimate of σ_u is close to zero. In fact, the implied estimate of ρ_{uv} is exactly -1. Moreover, the ML estimates are not the same as those implied by the ARIMA(2,2,3) model and the implied estimates lead to a structural error covariance matrix that is not positive definite. To see this, the estimate of ρ_{uv} implied by the ARIMA(2,2,3) estimates is -31.12.[9] These results indicate that double-drift UC model under the case II restrictions are not compatible with the data.

Figures 12 and 13 show the filtered and smoothed estimates of overall trend, irregular drift, and cycle from the double-drift model with $\sigma_{wu} = \sigma_{wv} = 0$. The estimated cycle is very close to the estimated cycle from Clark's double-drift UC0 model. The

[8]The medium unbiased estimate does not depend on the assumption of the correlation between trend and cycle shocks. In the equation (6), the contemporaneous correlation between the shock (v_t) to the smooth trend and the shock (w_t) to cycle does not change the dynamics of the error term (ζ_t).

[9]$\rho_{uv} = -0.1566/(0.0116 \times 0.4338) = -31.12$.

estimated drift, however, is similar in shape to estimated drift from Clark's model but shows much more variation due to the perfect negative correlation between the drift and cycle shocks.

5 Conclusion

This paper extends the results of MNZ to Clark's double-drift UC model that allows the growth rate of real GDP to follow a random walk. We show that the double-drift model with correlated components has an ARIMA(2,2,3) reduced form, and requires at least two restrictions for identification. Using the same postwar real GDP data as MNZ, we find that the double-drift model with uncorrelated shocks to the smooth trend and cycle produces results that are equivalent to those from an unrestricted ARIMA(2,2,3) model, and are similar to the results of MNZ. As in MNZ, we find evidence against the overidentification restriction implied by Clark's model with uncorrelated components. We further show that the double-drift model with uncorrelated shocks to the irregular drift and cycle is not supported by the data.

The primary purpose of our paper was to show that the logic and conclusions of MNZ hold in a univariate UC model with a flexible trend specification. Recently, several authors (e.g., Sinclair 2005 and Basistha 2005) have extended the framework of MNZ to multivariate models. In future research we plan to investigate the impact of the trend specification in multivariate UC models with correlated components.

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Appendix I: Beveridge-Nelson Decomposition for ARIMA(2,2,3) Process

The ARIMA(2,2,3) model (8) may be put in state space form with measurement equation

$$\Delta^2 y_t = \mathbf{Z}\ \boldsymbol{\alpha}_t$$

and transition equation

$$\boldsymbol{\alpha}_t = \mathbf{T}\ \boldsymbol{\alpha}_{t-1} + \mathbf{R}\ \epsilon_t,\ \ \epsilon_t \sim \text{iid } N(0, \sigma^2_\epsilon)$$

where $Z = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$ and

$$\boldsymbol{\alpha}_t = \begin{bmatrix} \Delta^2 y_t \\ \phi_2 \Delta^2 y_{t-1} + \theta_1 \epsilon_t + \theta_2 \epsilon_{t-1} + \theta_3 \epsilon_{t-2} \\ \theta_2 \epsilon_t + \theta_3 \epsilon_{t-1} \\ \theta_3 \epsilon_t \end{bmatrix},\ \mathbf{T} = \begin{bmatrix} \phi_1 & 1 & 0 & 0 \\ \phi_2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ \mathbf{R} = \begin{bmatrix} 1 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

By definition, the BN trend, BN_t, is the long-run forecast of y_t minus any deterministic components, DT_t:

$$\begin{aligned}
(1)\quad BN_t &= \lim_{J\to\infty} E_t[y_{t+J} - J\ DT_t] \\
&= \lim_{J\to\infty} E_t[y_t + \Delta y_{t+1} + \Delta y_{t+2} + \Delta y_{t+3} + \cdots + \Delta y_{t+J} - J\ DT_t] \\
&= \lim_{J\to\infty} E_t[y_t + \Delta y_t + \Delta^2 y_{t+1} \\
&\qquad\qquad + \Delta y_t + \Delta^2 y_{t+1} + \Delta^2 y_{t+2} \\
&\qquad\qquad + \Delta y_t + \Delta^2 y_{t+1} + \Delta^2 y_{t+2} + \Delta^2 y_{t+3} \\
&\qquad\qquad + \cdots \\
&\qquad\qquad + \Delta y_t + \Delta^2 y_{t+1} + \Delta^2 y_{t+2} + \cdots + \Delta^2 y_{t+J} - J\ DT_t] \\
&= y_t + \lim_{J\to\infty} E_t[J\Delta y_t + J\Delta^2 y_{t+1} + (J-1)\Delta^2 y_{t+2} + \cdots + \Delta^2 y_{t+J} - J\ DT_t]
\end{aligned}$$

Because the transition equation of the state space representation is a difference equation, $E_t[\Delta^2 y_{t+i}]$ may be expressed in terms of $\mathbf{T}$ and $\boldsymbol{\alpha}_t$, where the eigenvalues of $\mathbf{T}$ are less than one in modulus, as follows:

$$(2) \quad E_t[\Delta^2 y_{t+i}] = E_t[\boldsymbol{Z\alpha}_{t+i}] = \mathbf{ZT}^i \boldsymbol{\alpha}_{t|t}$$

where $\alpha_{t|t} = E_t(\alpha_t)$ is the filtered estimator of α_t from the Kalman filter. By substituting (2) into (1), BN_t may be reexpressed as

$$\begin{aligned}(3) \quad BN_t &= y_t + \lim_{J\to\infty} J(\Delta y_t - DT_t) \\ &+ \lim_{J\to\infty} \{J\boldsymbol{ZT\alpha}_{t|t} + (J-1)\mathbf{ZT}^2\boldsymbol{\alpha}_{t|t} \\ &+ \cdots + \mathbf{ZT}^J\boldsymbol{\alpha}_{t|t}\} \\ &= y_t + \lim_{J\to\infty} J(\Delta y_t - DT_t) + \lim_{J\to\infty} \mathbf{S}_J \ \boldsymbol{\alpha}_{t|t}\end{aligned}$$

where $\mathbf{S}_J = J\mathbf{ZT} + (J-1)\mathbf{ZT}^2 + \cdots + \mathbf{ZT}^J$. The expression for $\mathbf{S}_J$ may be simplified as follows. First multiply $\mathbf{S}_J$ by $\mathbf{T}$ and then subtract $\mathbf{S}_J$ giving

$$\begin{aligned}\mathbf{S}_J(\mathbf{T} - \mathbf{I}_4) &= \mathbf{ZT}^2 + \mathbf{ZT}^3 + \cdots + \mathbf{ZT}^{J+1} - J\mathbf{ZT} \\ &= \mathbf{Z}\ (\mathbf{T}^2 + \mathbf{T}^3 + \cdots + \mathbf{T}^{J+1}) - J\mathbf{ZT} \\ &= \mathbf{Z}\ \mathbf{T}^2(\mathbf{I}_4 - \mathbf{T}^J)(\mathbf{I}_4 - \mathbf{T})^{-1} - J\mathbf{ZT}\end{aligned}$$

Then, multiply by $-(\mathbf{I}_4 - \mathbf{T})^{-1}$ to solve for $\mathbf{S}_J$ giving

$$(4) \quad \mathbf{S}_J = -\mathbf{Z}\ \mathbf{T}^2(\mathbf{I}_4 - \mathbf{T}^J)(\mathbf{I}_4 - \mathbf{T})^{-2} + J\mathbf{ZT}(\mathbf{I}_4 - \mathbf{T})^{-1}$$

Substituting (4) for $\mathbf{S}_J$ in (3) then gives

$$\begin{aligned}(5) \quad BN_t &= y_t + \lim_{J\to\infty} J(\Delta y_t - DT_t) \\ &+ \lim_{J\to\infty} \{-\mathbf{ZT}^2(\mathbf{I}_4 - \mathbf{T}^J)(\mathbf{I}_4 - \mathbf{T})^{-2} + J\mathbf{ZT}(\mathbf{I}_4 - \mathbf{T})^{-1}\}\boldsymbol{\alpha}_{t|t} \\ &= y_t + \lim_{J\to\infty} J\{\Delta y_t + \mathbf{ZT}(\mathbf{I}_4 - \mathbf{T})^{-1}\boldsymbol{\alpha}_{t|t} - DT_t\} - \mathbf{ZT}^2(\mathbf{I}_4 - \mathbf{T})^{-2}\boldsymbol{\alpha}_{t|t}\end{aligned}$$

For BN_t to be finite, define the deterministic drift as

$$DT_t = \Delta y_t + \mathbf{ZT}(\mathbf{I}_4 - \mathbf{T})^{-1}\boldsymbol{\alpha}_{t|t}$$

Adopting DT_t defined above produces

$$BN_t = y_t - \mathbf{ZT}^2(\mathbf{I}_4 - \mathbf{T})^{-2}\boldsymbol{\alpha}_{t|t}$$

The BN cycle may then be defined as the difference between y_t and BN_t:

$$\begin{aligned} C_t^{BN} &= y_t - BN_t \\ &= \mathbf{ZT}^2(\mathbf{I}_4 - \mathbf{T})^{-2}\boldsymbol{\alpha}_{t|t} \end{aligned}$$

Appendix II: Numerical Optimization

1. **Optimization Procedure**

   Many of the estimation results in the paper depend on the initial values for the optimizer. To avoid getting stuck at potential local maxima, we used a genetic algorithm (Matlab program gagordy.m) to find adequate initial values. Genetic algorithms (GAs) are based on a biological metaphor. Likelihood values ("fitness" in GA terminology) of candidates contribute to choose the next generation of candidates without considering the curvature of the likelihood function. An attractive feature of GAs is that they can handle local optimum because they searches various regions of the parameter space by generating some candidates randomly, which are set by a "mutation ratio." However, since GAs take a long time to converge with predefined criteria, we only used them to find an adequate initial value by setting a loose criteria.

2. **Parameter Restriction**

   Here we extend the parameter constraints used in Kim and Nelson (1999, Section 2.3.1) to force the $AR(2)$ cycle component to be stationary. In their derivation, they imposed the restriction that the roots of the AR(2) polynomial are real. Here, we remove this restriction and allow the roots to be complex.

   - **Parameter Restriction for Stationarity of AR(2)**

     For the AR(2) polynomial to have complex number roots, it is written as follows:

$$(z-(a+bi))(z-(a-bi)) = z^2 - 2az + (a^2+b^2)$$

where $a^2 + b^2 < 1$ for the modulus of roots to be within unit circle. The parameters of the AR(2) are given as

$$\phi_1 = 2a, \quad \phi_2 = -(a^2 + b^2)$$

Let $z_1 = \frac{x_1^{uc}}{1+|x_1^{uc}|}, \quad z_2 = \frac{|x_2^{uc}|}{1+|x_2^{uc}|}$ where x_i^{uc} is an unrestricted real number. To satisfy the constraint of $a^2 + b^2 < 1$, we write $a = z_1,\ b^2 = z_2(1 - z_1^2)$. The relations for the AR(2) parameters are then

$$\phi_1 = 2z_1$$

$$\phi_2 = -(z_1^2 + z_2(1 - z_1^2))$$

- **Parameter Restriction for Invertibility of MA(3)**

For the MA(3) polynomial to have complex number roots, it is written as follows:

$$(z-(a+bi))(z-(a-bi))(z-c) = z^3-(2a+c)z^2+(a^2+b^2+2ac)z-c(a^2+b^2)$$

where $a^2 + b^2 < 1$ and $|c| < 1$ for the modulus of roots to be within unit circle. The parameters of the MA(3) are given as

$$\theta_1 = -(2a + c), \quad \theta_2 = (a^2 + b^2 + 2ac), \quad \theta_3 = -c(a^2 + b^2)$$

Let $z_1 = \frac{x_1^{uc}}{1+|x_1^{uc}|}, \quad z_2 = \frac{|x_2^{uc}|}{1+|x_2^{uc}|}, \quad z_3 = \frac{x_3^{uc}}{1+|x_3^{uc}|}$ where x_i^{uc} is an unrestricted real number. To satisfy the constraint of $a^2 + b^2 < 1$ and $|c| < 1$, we write $a = z_1,\ b^2 = z_2(1 - z_1^2),\ c = z_3$. The relations for the MA(3) parameters are then

$$\theta_1 = -(2z_1 + z_3)$$

$$\theta_2 = +(z_1^2 + z_2(1 - z_1^2) + 2z_1z_3)$$

$$\theta_3 = -z_3(z_1^2 + z_2(1 - z_1^2))$$

Table 1: Maximum Likelihood Estimates for UC0 Model

	MNZ UC0 model[1]		Clark UC0 model[1]	
	estimate	standard error	estimate	standard error
d	0.8119	(0.0501)	-	-
ϕ_1	1.5303	(0.1019)	1.5007	(0.1177)
ϕ_2	-0.6097	(0.1150)	-0.5877	(0.1284)
σ_w	0.6893	(0.1039)	0.6423	(0.1324)
σ_u	-	-	0.0199	(0.0127)
σ_v	0.6199	(0.1320)	0.6567	(0.1514)
Log likelihood	-285.3815		-287.1492	
AR roots	$1.2550 \pm 0.2555i$		$1.2769 \pm 0.2670i$	

1. Estimation sample: 1947:I - 1998:II (# 206), log likelihood sample: 1947:III - 1998:II (# 204)

Table 2: Maximum Likelihood Estimates for ARIMA(4,2,5)

ϕ_1	ϕ_2	ϕ_3	ϕ_4	σ_ϵ^2
-0.3398	-0.1382	-0.2503	-0.0668	0.0101
(0.5552)	(0.4626)	(0.3655)	(0.2687)	
θ_1	θ_2	θ_3	θ_4	θ_5
-0.2789	-0.1928	-0.0236	-0.2725	-0.2239
(0.5491)	(0.6255)	(0.5125)	(0.4056)	(0.1876)
Inverted AR Roots :	$0.25 \pm 0.58i, \; -0.32, \; -0.52$			
Inverted MA Roots :	$1, \; 0.20 \pm 0.74i, \; -0.56 \pm 0.27i$			

* Standard errors are in parentheses. Estimation was performed using Eviews 5.1.

Table 3: UC model parameters implied by ARIMA(2,2,3) model

	Case I UC Model	**Case II UC Model**
ϕ_1	1.4400	1.4400
ϕ_2	-0.6200	-0.620
σ_w	0.5394	0.6871
σ_u	0.1089	0.1089
σ_v	0.4181	0.5252
σ_{wu}	0	0
σ_{wv}	0.1737	0
σ_{uv}	0	0.0156

Table 4: Maximum Likelihood Estimates for ARIMA(2,2,3)

	Estimate	Standard Error
ϕ_1	1.3368	(0.1502)
ϕ_2	-0.7006	(0.1521)
θ_1	-2.0379	(0.1844)
θ_2	1.5518	(0.3487)
θ_3	-0.5095	(0.1902)
σ_ϵ	0.9748	(0.0485)
Log likelihood	-286.125	
AR roots	$0.9540 \pm 0.7191i$	
MA roots	$1.0093 + 2.6883e - 24i, 1.0181 \pm 9.5286e - 1i$	

* Estimation sample: 1947:III - 1998:II (# 204); log likelihood sample: 1947:III - 1998:II (# 204)

Table 5: Maximum Likelihood Estimates for ARIMA(2,1,2)

	ARIMA(2,1,2)		ARIMA(2,2,3)
	Estimate	Standard Error	Implied by ARIMA(2,1,2)
μ	0.8137	(0.0867)	-
ϕ_1	1.3418	(0.1450)	1.3418
ϕ_2	-0.7057	(0.1492)	-0.7057
θ_1	-1.0541	(0.1803)	-2.0541
θ_2	0.5183	(0.1931)	1.5725
θ_3	-	-	-0.5183
σ_ϵ	0.9694	(0.0479)	0.9694
Log likelihood	-283.43		
AR roots	$0.9507691 \pm 0.7163495i$		
MA roots	$1.016857 \pm 0.9461868i$		

* Estimation sample: 1947:II - 1998:II (# 205); log likelihood sample: 1947:III - 1998:II (# 204)

Table 6: Double-Drift UC Model with $\sigma_{wu} = \sigma_{uv} = 0$

	UC model implied by		UC model[3] (direct estimation)	
	ARIMA(2,2,3)[1]	ARIMA(2,1,2)[2]	estimate	(standard error)
ϕ_1	1.3368	1.3418	1.3368	(0.1484)
ϕ_2	-0.7006	-0.7057	-0.7007	(0.1470)
σ_w	1.2458	1.2562	1.2458	(0.1704)
σ_u	0.0116	0.0000	0.0116	(0.0228)
σ_v	0.7613	0.7609	0.7613	(0.2895)
σ_{wv}	-0.8610	-0.8660	-0.8610	(0.4269)
Log likelihood	-286.125	-283.43	-286.125	

1. Estimation sample: 1947:III - 1998:II (# 204), log likelihood sample: 1947:III - 1998:II (# 204)
2. Estimation sample: 1947:II - 1998:II (# 205), log likelihood sample: 1947:III - 1998:II (# 204)
3. Estimation sample: 1947:I - 1998:II (# 206), log likelihood sample: 1947:III - 1998:II (# 204)

Table 7: ML Estimates for Double-Drift UC Model

with $\sigma_{wu} = \sigma_{uv} = 0$ and $\sigma_u = 0.0036$

	Estimate	Standard Error
ϕ_1	1.3383	(0.1549)
ϕ_2	-0.7024	(0.1690)
σ_w	1.2385	(0.1715)
σ_v	0.7540	(0.3059)
σ_{wv}	-0.8450	(0.4191)
Log likelihood	-286.6655	
AR roots	$0.9525829 \pm 0.7184735i$	

* Estimation sample: 1947:I - 1998:II (# 206); log likelihood sample: 1947:III - 1998:II (# 204)

Table 8: Double-Drift UC Model with $\sigma_{wu} = \sigma_{wv} = 0$

	UC model implied by		UC model[3] (direct estimation)	
	ARIMA(2,2,3)[1]	ARIMA(2,1,2)[2]	(estimate)	(standard error)
ϕ_1	1.3368	1.3418	1.4968	(0.1142)
ϕ_2	-0.7006	-0.7057	-0.6103	(0.1283)
σ_w	0.8313	0.8438	0.6576	(0.1241)
σ_u	0.0116	0.0000	0.0200	(0.0145)
σ_v	0.4338	0.4365	0.6200	(0.1501)
σ_{uv}	-0.1566	-0.1575	-0.0124	(0.0089)
Log likelihood	-286.125	-284.6498	-288.9136	

1. Estimation sample: 1947:III - 1998:II (# 204), log likelihood sample: 1947:III - 1998:II (# 204)
2. Estimation sample: 1947:II - 1998:II (# 205), log likelihood sample: 1947:III - 1998:II (# 204)
3. Estimation sample: 1947:I - 1998:II (# 206), log likelihood sample: 1947:III - 1998:II (# 204)

Figure 1: Clark UC0 Double-Drift Model: Filtered and Smoothed Drift

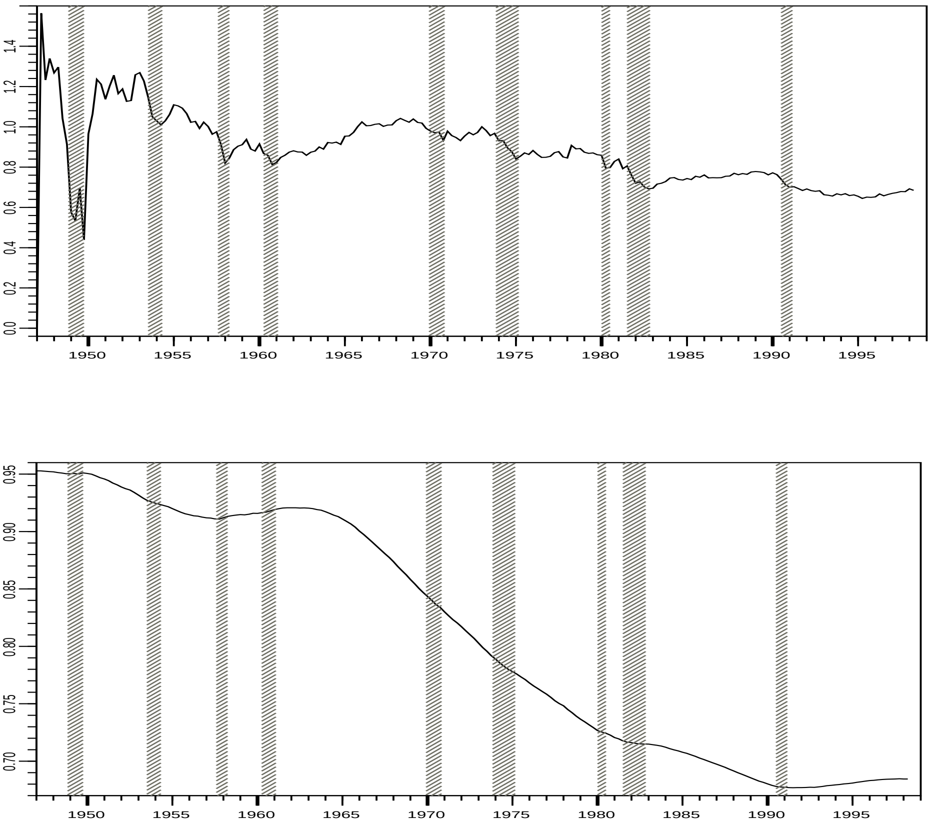

Figure 2: MNZ UC0 and Clark UC0 (Filtered and Smoothed Cycles)

MNZ UC0: —— , Clark UC0: - - -

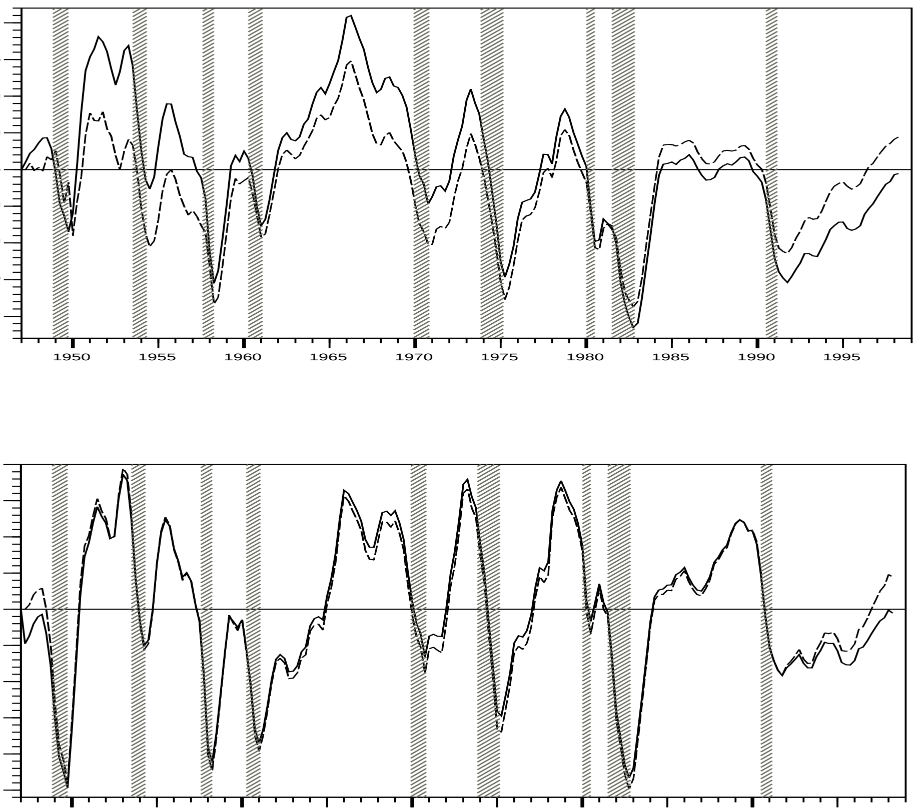

Figure 3: Clark model : Filtered Trend, Irregular Trend, and Cycle

Using Median-Unbiased Estimate of $\sigma_u = 0.036$

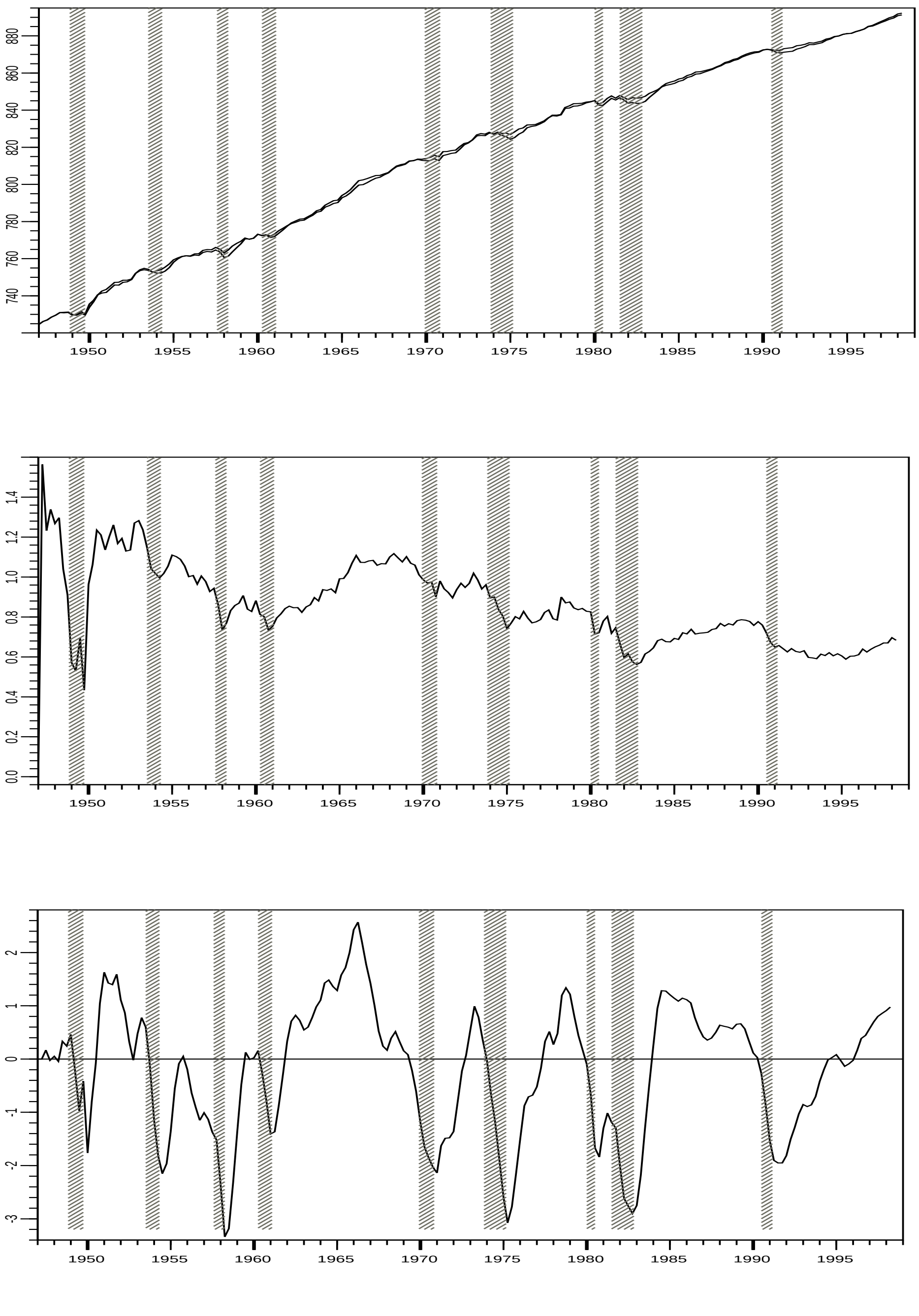

Figure 4: Clark model : Smoothed Trend, Irregular Trend, and Cycle

Using Median-Unbiased Estimate of $\sigma_u = 0.036$

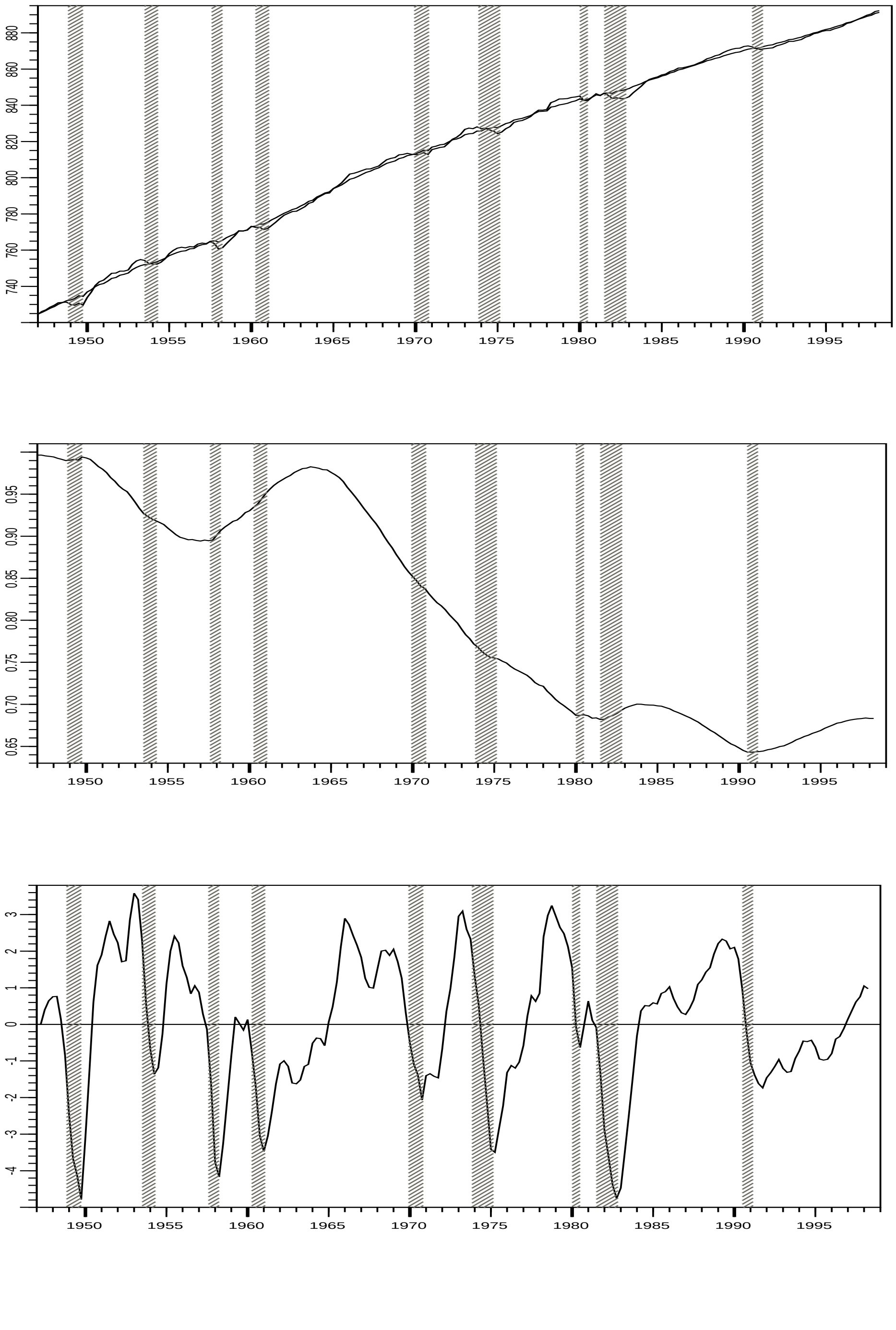

Figure 5: Beveridge-Nelson Decomposition

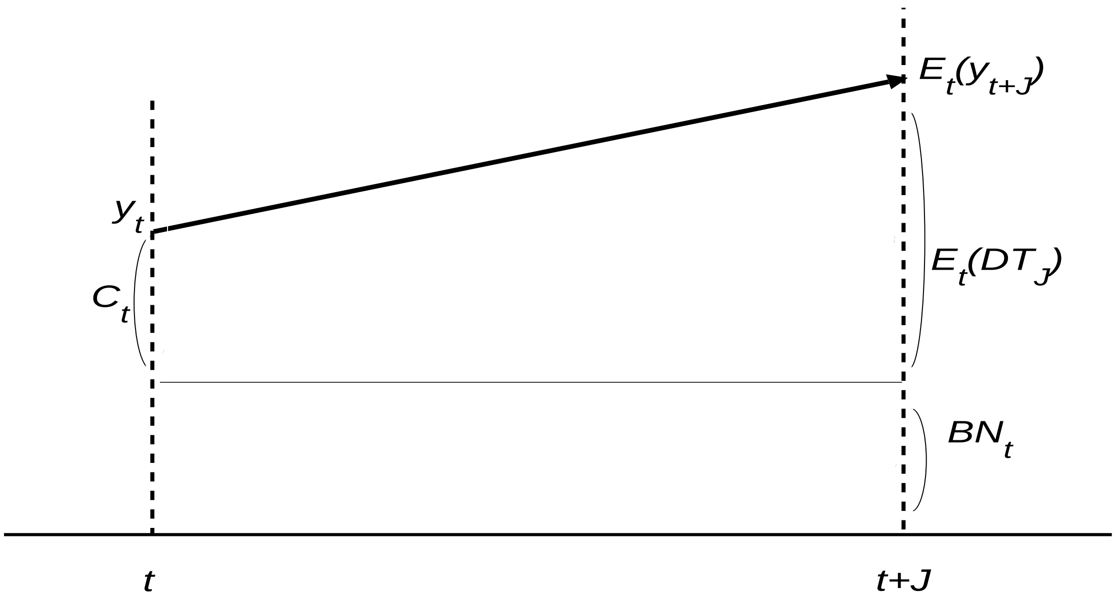

Figure 6: Cycles of Beveridge-Nelson Decomposition and UC-UR(2) for ARIMA(2,2,3)

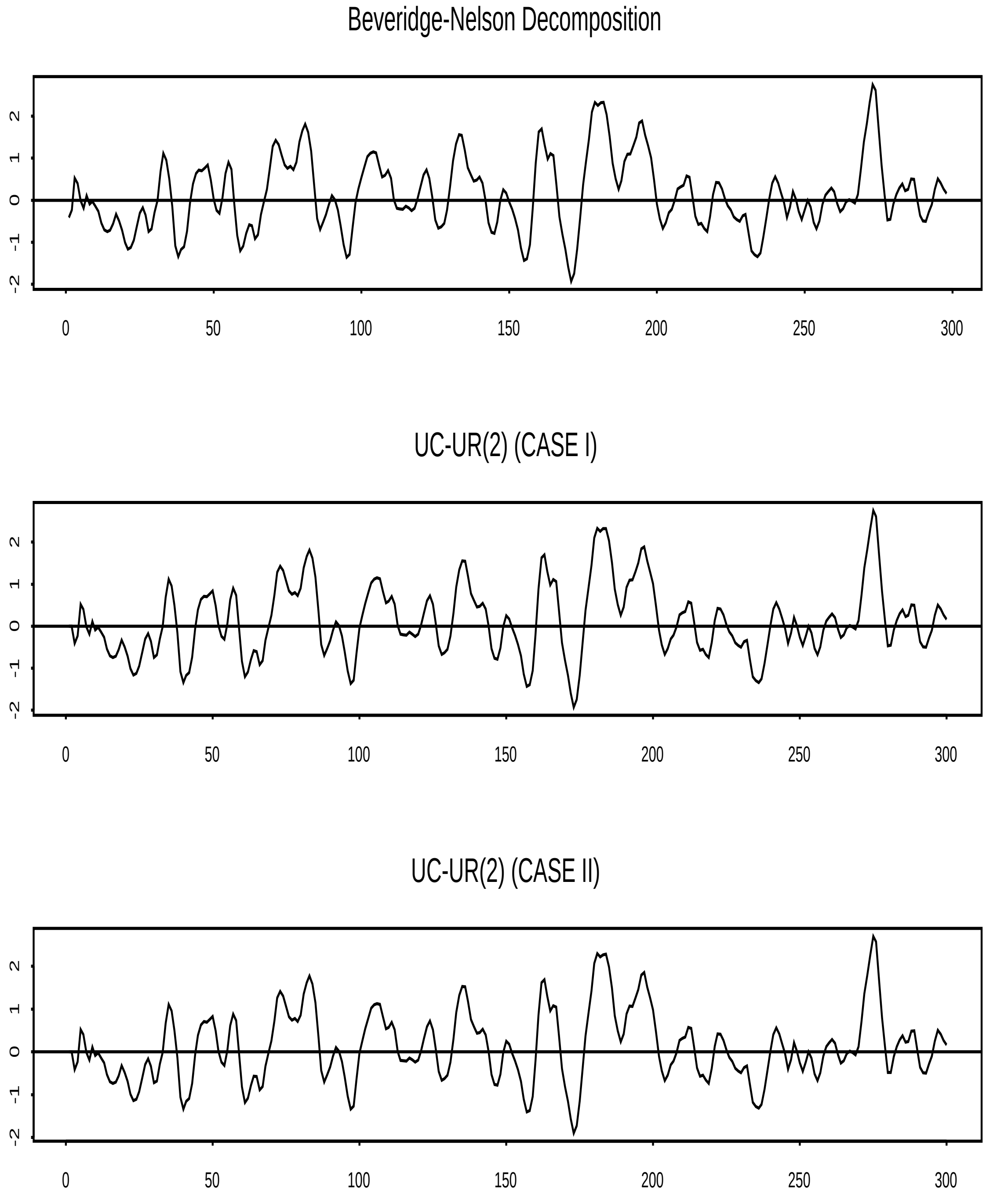

Figure 7: Beveridge-Nelson Decomposition for ARIMA(2,2,3)

(Trend, Irregular Trend, and Cycle)

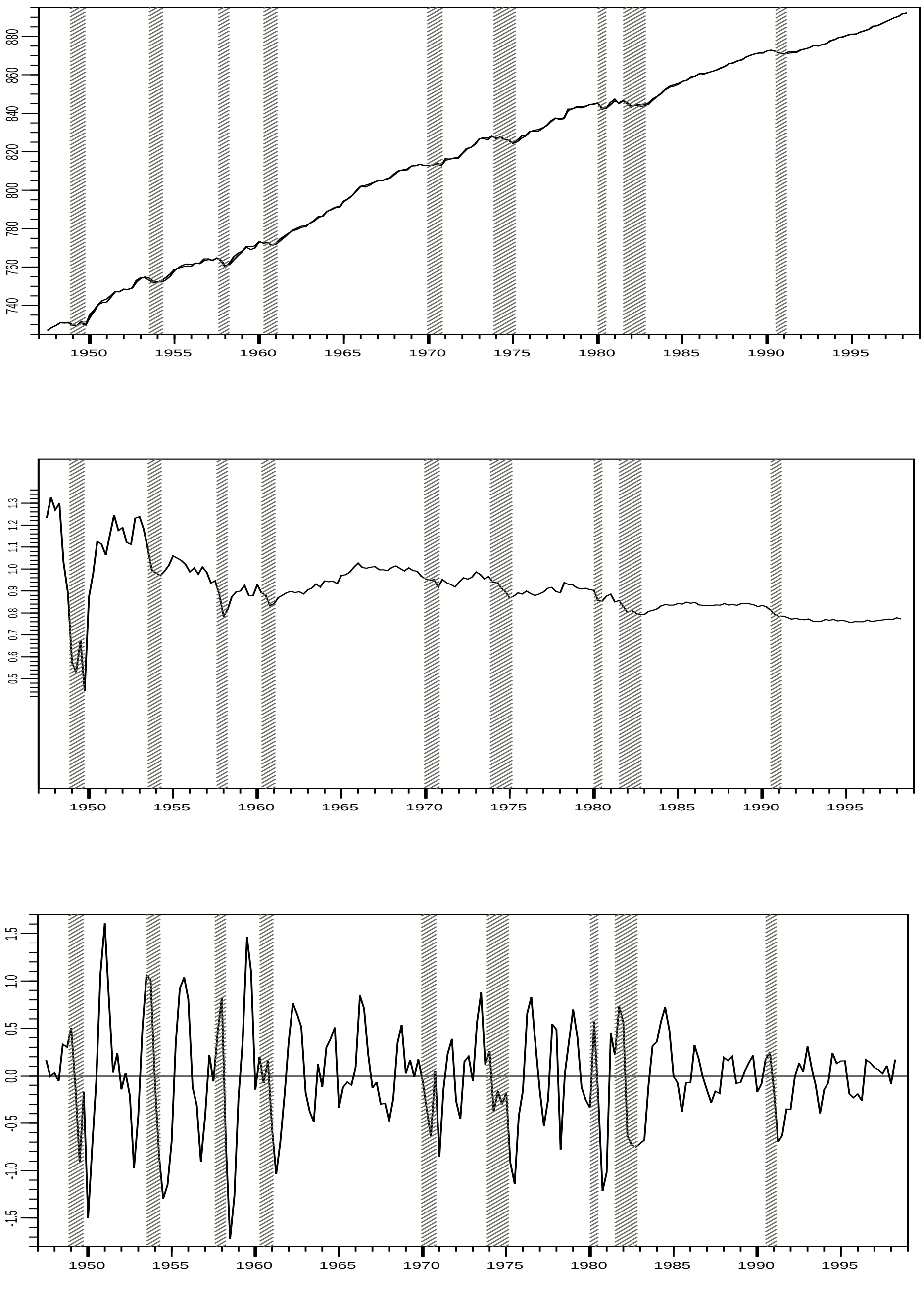

Figure 8: UC-UR(2) with Restrictions of $\sigma_{wu} = \sigma_{uv} = 0$

(Filtered Trend, Irregular Trend, and Cycle)

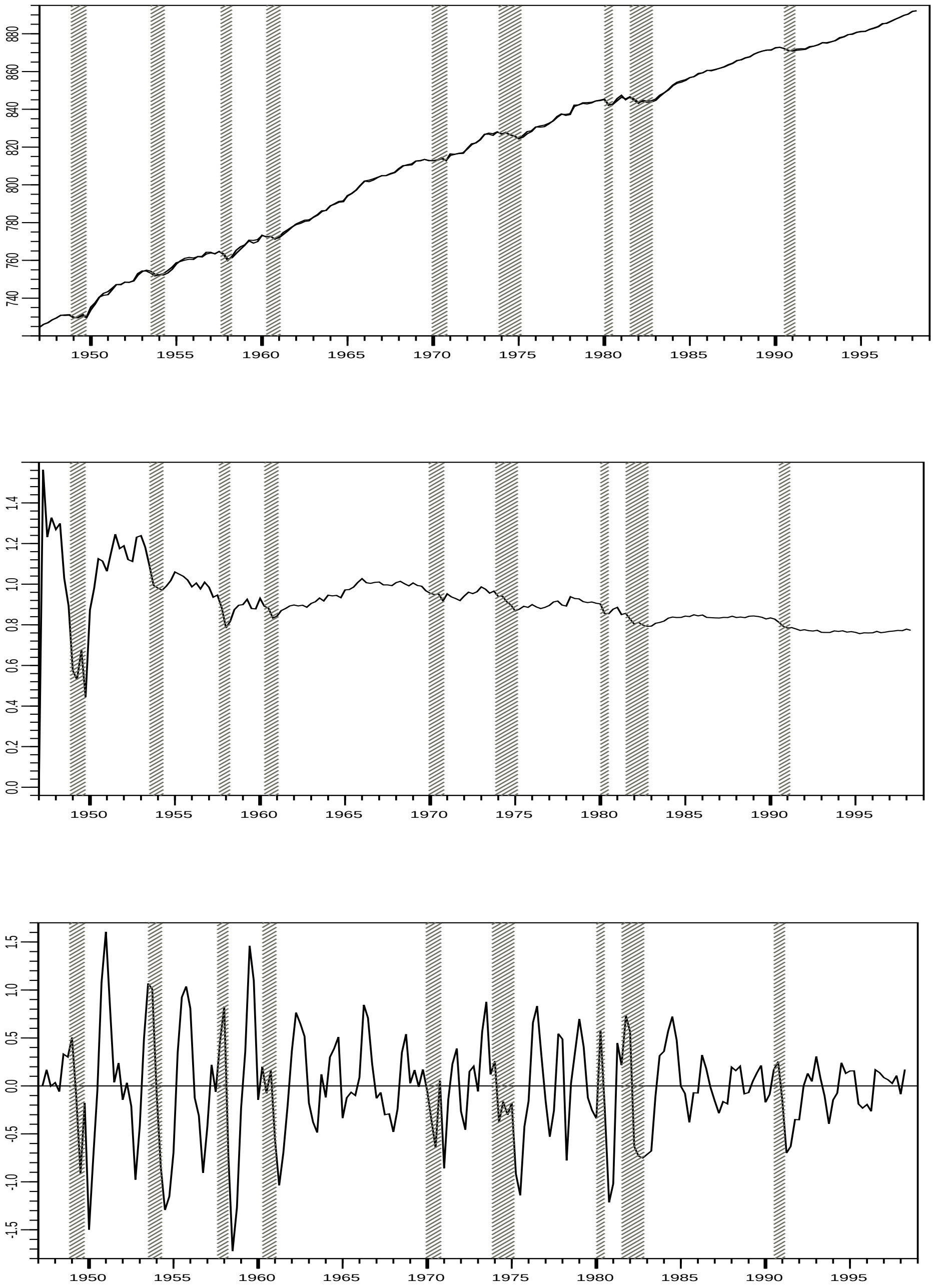

Figure 9: UC-UR(2) with Restrictions of $\sigma_{wu} = \sigma_{uv} = 0$

(Smoothed Trend, Irregular Trend, and Cycle)

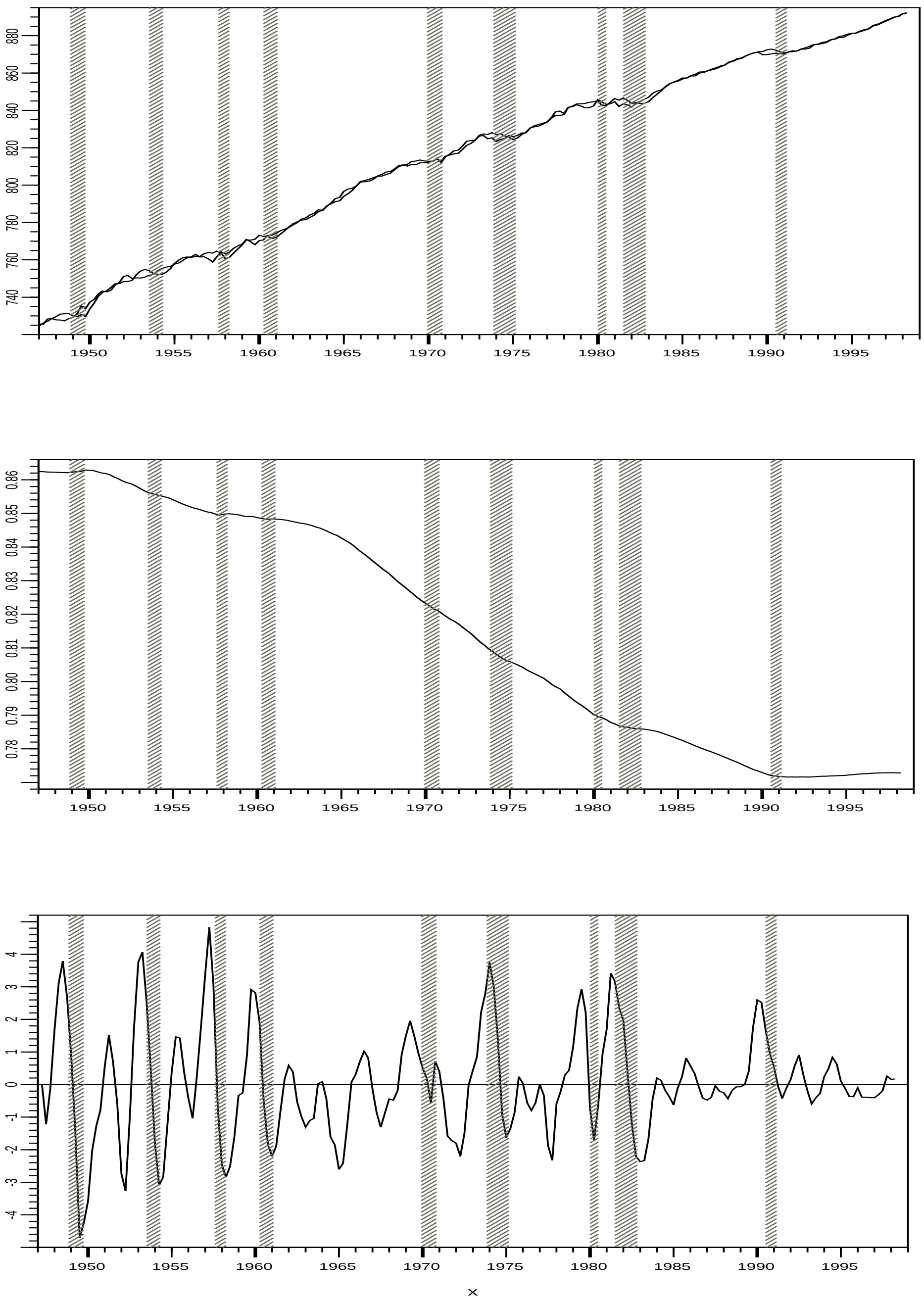

Figure 10: Double-Drift UC model with with Restrictions of $\sigma_{wu} = \sigma_{uv} = 0$, and $\sigma_u = 0.036$

Filtered Trend, Irregular Trend, and Cycle

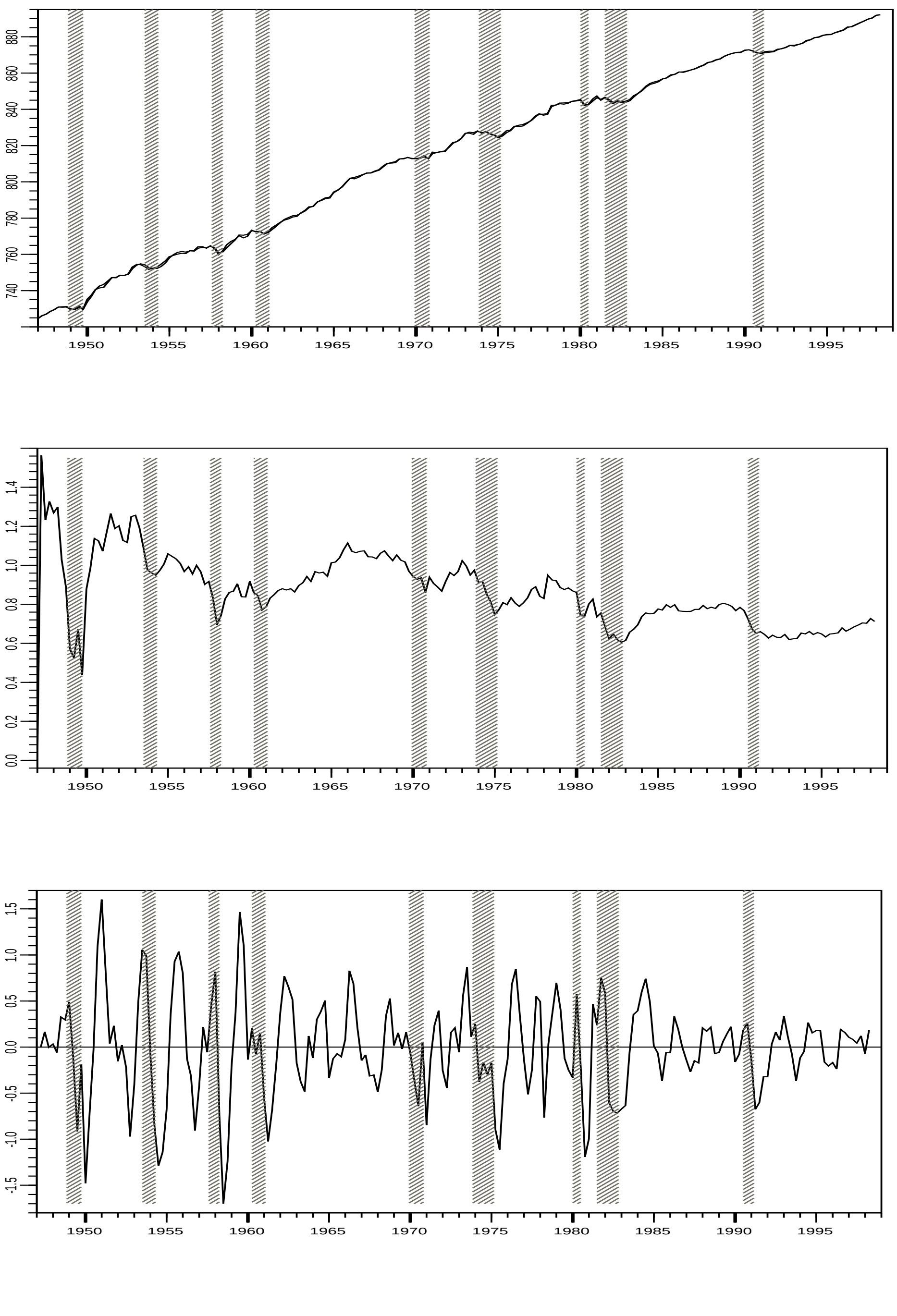

Figure 11: Double-Drift UC model with with Restrictions of $\sigma_{wu} = \sigma_{uv} = 0$, and $\sigma_u = 0.036$

Smoothed Trend, Irregular Trend, and Cycle

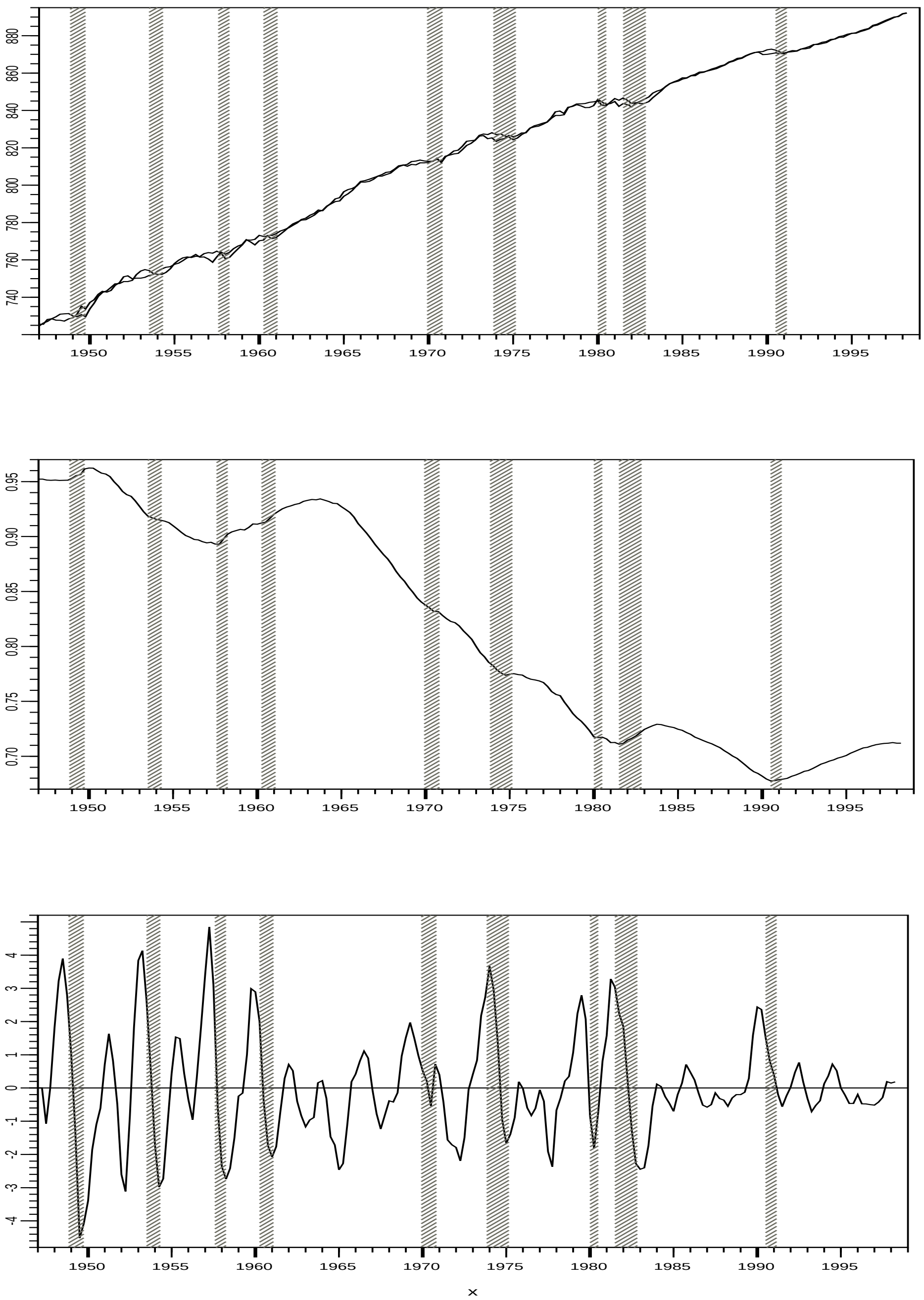

Figure 12: UC-UR(2) with Restrictions of $\sigma_{wu} = \sigma_{wv} = 0$

(Filtered Trend, Irregular Trend, and Cycle)

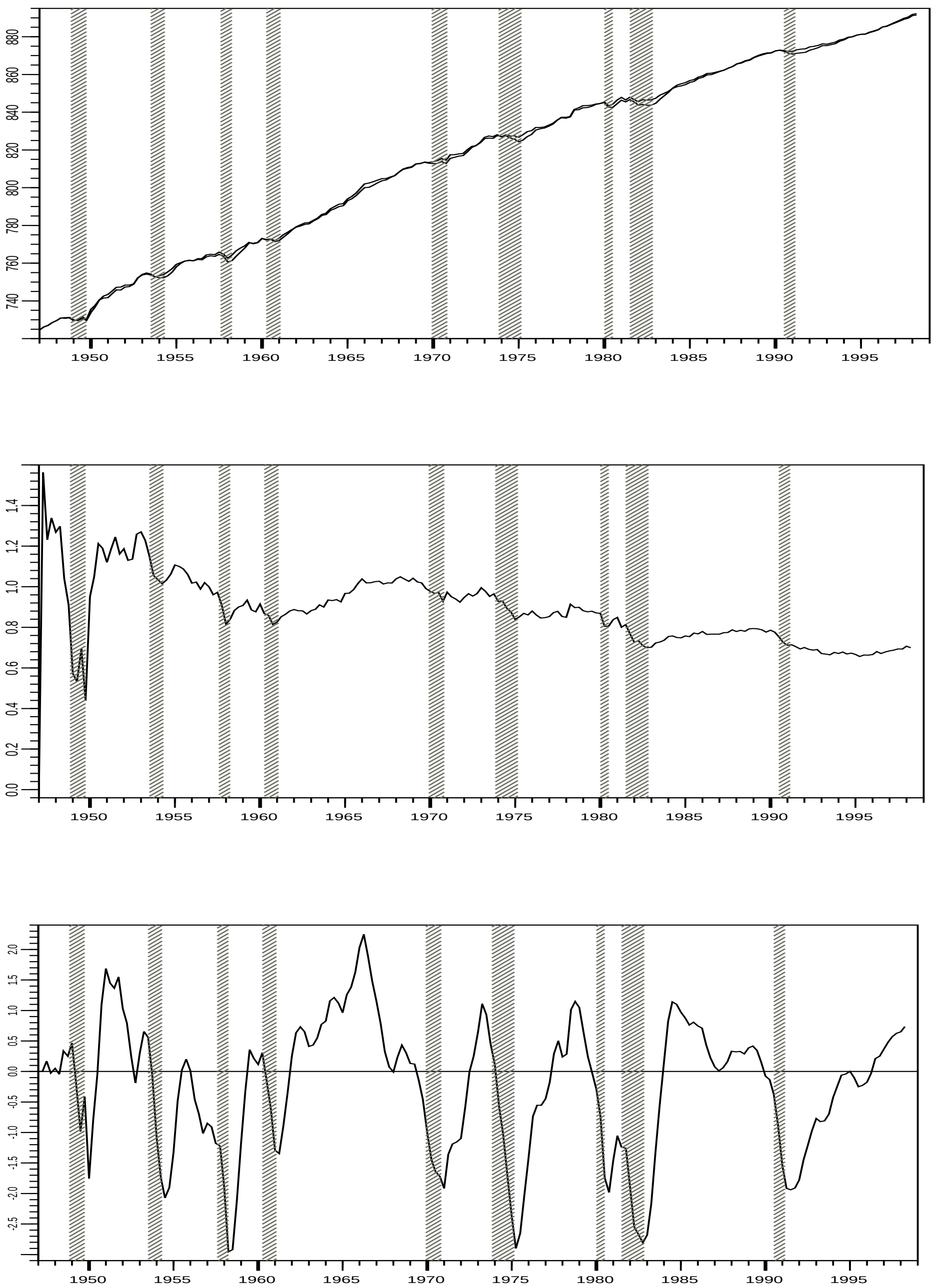

Figure 13: UC-UR(2) with Restrictions of $\sigma_{wu} = \sigma_{wv} = 0$

(Smoothed Trend, Irregular Trend, and Cycle)