Factor Model Based Risk Measurement and Management

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Risk Measurement and Management

- Quantify asset and portfolio exposures to risk factors
  - Equity, rates, credit, volatility, currency
  - Style, geography, industry,
- Quantify asset and portfolio risk
  - SD, VaR, ETL
- Perform risk decomposition
  - Contribution of risk factors, contribution of constituent assets to portfolio risk
- Stress testing and scenario analysis
Asset Level Linear Factor Model

\[ R_{it} = \alpha_i + \beta_{i1} F_{1t} + \cdots + \beta_{ik} F_{kt} + \varepsilon_{it}, \]
\[ = \alpha_i + \beta_i' F_t + \varepsilon_{it} \]
\[ i = 1, \ldots, n; \ t = t_i, \ldots, T \]

\[ F_t \sim (\mu_F, \Sigma_F) \]
\[ \varepsilon_{it} \sim (0, \sigma^2_{\varepsilon,i}) \]
\[ \text{cov}(F_{jt}, \varepsilon_{is}) = 0 \text{ for all } j, i, s \text{ and } t \]
\[ \text{cov}(\varepsilon_{it}, \varepsilon_{js}) = 0 \text{ for } i \neq j, s \text{ and } t \]

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Performance Attribution

\[ E[R_{it}] = \alpha_i + \beta_{i1} E[F_{1t}] + \cdots + \beta_{ik} E[F_{kt}] \]

Expected return due to systematic “beta” exposure

\[ \beta_{i1} E[F_{1t}] + \cdots + \beta_{ik} E[F_{kt}] \]

Expected return due to firm specific “alpha”

\[ \alpha_i = E[R_{it}] - (\beta_{i1} E[F_{1t}] + \cdots + \beta_{ik} E[F_{kt}]) \]
Factor Model Covariance

\[
R_t = \alpha + BF_t + \varepsilon_t
\]

\[
\text{var}(R_t) = \Sigma_{FM} = B\Sigma_F B' + D_\varepsilon
\]

\[
D_\varepsilon = \text{diag}(\sigma_{\varepsilon,1}^2, \ldots, \sigma_{\varepsilon,n}^2)
\]

Note:

\[
\text{cov}(R_{it}, R_{jt}) = \beta_i' \text{var}(F_t) \beta_j = \beta_i' \Sigma_F \beta_j
\]

\[
\text{var}(R_{it}) = \beta_i' \Sigma_F \beta_i + \sigma_{\varepsilon,i}^2
\]
Portfolio Linear Factor Model

\[ w = (w_1, \ldots, w_n)' = \text{portfolio weights} \]

\[ \sum_{i=1}^{n} w_i = 1, \; w_i \geq 0 \text{ for } i = 1, \ldots, n \]

\[ R_{p,t} = w'R_t = w'\alpha + w'BF_t + w'e_t \]

\[ = \sum_{i=1}^{n} w_i R_{it} = \sum_{i=1}^{n} w_i \alpha_i + \sum_{i=1}^{n} w_i \beta_i'F_t + \sum_{i=1}^{n} w_i \epsilon_{it} \]

\[ = \alpha_p + \beta_p'F_t + \epsilon_{p,t} \]
Risk Measures

Return Standard Deviation (SD, aka active risk)

\[ \sigma = SD(R_t) = \left( \beta' \Sigma_F \beta + \sigma^2_{\varepsilon} \right)^{1/2} \]

Value-at-Risk (VaR)

\[ VaR_\alpha = q_\alpha = F^{-1}(\alpha), \ 0.01 \leq \alpha \leq 0.10 \]

\[ F = \text{CDF of return } R_t \]

Expected Tail Loss (ETL)

\[ ETL_\alpha = E[R_t \mid R_t \leq VaR_\alpha] \]

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Risk Measures

Returns

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Tail Risk Measures: Non-Normal Distributions

- Asset returns are typically non-normal
- Many possible univariate non-normal distributions
  - Student’s-$t$, skewed-$t$, generalized hyperbolic, Gram-Charlier, $\alpha$-stable, generalized Pareto, etc.
- Need multivariate non-normal distributions for portfolio analysis and risk budgeting.
- Large number of assets, small samples and unequal histories make multivariate modeling difficult
Factor Model Monte Carlo (FMMC)

- Use fitted factor model to simulate pseudo asset return data preserving empirical characteristics of risk factors and residuals
  - Use *full data* for factors and *unequal history* for assets to deal with missing data
- Estimate tail risk and related measures non-parametrically from simulated return data
Simulation Algorithm

- Simulate $B$ values of the risk factors by re-sampling from *full sample* empirical distribution:
  \[
  \{ \mathbf{F}_{1}^{*}, \ldots, \mathbf{F}_{B}^{*} \}
  \]

- Simulate $B$ values of the factor model residuals from fitted non-normal distribution:
  \[
  \{ \hat{\mathbf{e}}_{i1}^{*}, \ldots, \hat{\mathbf{e}}_{iB}^{*} \}, \quad i = 1, \ldots, n
  \]

- Create factor model returns from factor models fit over *truncated samples*, simulated factor variables drawn from *full sample* and simulated residuals:
  \[
  R_{it}^{*} = \hat{\alpha}_i + \hat{\beta}_i \mathbf{F}_t^{*} + \hat{\mathbf{e}}_{it}^{*}, \quad t = 1, \ldots, B; i = 1, \ldots, n
  \]
What to do with $\{R_{it}^*, F_{it}^*, \hat{E}_{it}^*\}_{t=1}^B$?

- Backfill missing asset performance
- Compute asset and portfolio performance measures (e.g., Sharpe ratios)
- Compute *non-parametric* estimates of asset and portfolio tail risk measures
- Compute *non-parametric* estimates of asset and factor contributions to portfolio tail risk measures
Factor Risk Budgeting

Given linear factor model for asset or portfolio returns,

\[ R_t = \alpha + \beta' F_t + \varepsilon_t = \alpha + \beta' F_t + \sigma \varepsilon \times z_t = \alpha + \tilde{\beta}' \tilde{F}_t \]

\[ \tilde{\beta}' = (\beta', \sigma \varepsilon), \quad \tilde{F}_t = (F'_t, z_t)', \quad z_t \sim (0, 1) \]

SD, VaR and ETL are linearly homogenous functions of factor sensitivities \( \tilde{\beta} \). Euler’s theorem gives additive decomposition

\[ RM(\tilde{\beta}) = \sum_{j=1}^{k+1} \tilde{\beta}_j \frac{\partial RM(\tilde{\beta})}{\partial \tilde{\beta}_j}, \quad RM = SD, \ VaR_\alpha, \ ETL_\alpha \]
Factor Contributions to Risk

Marginal Contribution to Risk of factor $j$:

$$\tilde{\beta}_j \frac{\partial RM (\tilde{\beta})}{\partial \tilde{\beta}_j}$$

Contribution to Risk of factor $j$:

$$\tilde{\beta}_j \frac{\partial RM (\tilde{\beta})}{\partial \tilde{\beta}_j}$$

Percent Contribution to Risk of factor $j$:

$$\tilde{\beta}_j \frac{\partial RM (\tilde{\beta})}{\partial \tilde{\beta}_j} / RM (\tilde{\beta})$$
Factor Tail Risk Contributions

For $RM = VaR, ETL$ it can be shown that

$$\frac{\partial VaR_{\alpha}(\tilde{\beta})}{\partial \beta_j} = E[\tilde{F}_{jt} \mid R_t = VaR_{\alpha}], \ j = 1, \ldots, k + 1$$

$$\frac{\partial ETL_{\alpha}(\tilde{\beta})}{\partial \beta_j} = E[\tilde{F}_{jt} \mid R_t \leq VaR_{\alpha}], \ j = 1, \ldots, k + 1$$

Notes:
1. Intuitive interpretations as stress loss scenarios
2. Analytic results are available under normality

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Semi-Parametric Estimation

Factor Model Monte Carlo semi-parametric estimates

\[
\hat{E}[\tilde{F}_{jt} | R_t = \text{VaR}_\alpha] = \frac{1}{m} \sum_{t=1}^{B} \tilde{F}_{jt}^* \cdot 1\left\{ \text{VaR}_\alpha - \epsilon \leq R_t^* \leq \text{VaR}_\alpha + \epsilon \right\}
\]

\[
\hat{E}[\tilde{F}_{jt} | R_t \leq \text{VaR}_\alpha] = \frac{1}{[B\alpha]} \sum_{t=1}^{B} \tilde{F}_{jt}^* \cdot 1\left\{ R_t^* \leq \text{VaR}_\alpha \right\}
\]

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Hedge fund returns and 5% VaR Violations

Risk factor returns when fund return <= 5% VaR

Factor marginal contribution to 5% ETL
Given portfolio returns,

$$R_{p,t} = \mathbf{w}'\mathbf{R}_t = \sum_{i=1}^{n} w_i R_{it}$$

$SD$, $VaR$ and $ETL$ are linearly homogenous functions of portfolio weights $\mathbf{w}$. Euler’s theorem gives additive decomposition

$$RM\left(\mathbf{w}\right) = \sum_{i=1}^{n} w_i \frac{\partial RM\left(\mathbf{w}\right)}{\partial w_i}, \quad RM = SD, \ VaR_\alpha, \ ETL_\alpha$$
Marginal Contribution to Risk of asset $i$: \[ \frac{\partial RM (w)}{\partial w_i} \]

Contribution to Risk of asset $i$: \[ w_i \frac{\partial RM (w)}{\partial w_i} \]

Percent Contribution to Risk of asset $i$: \[ \frac{w_i \frac{\partial RM (w)}{\partial w_i}}{RM (w)} \]
Portfolio Tail Risk Contributions

For $RM = VaR$, $ETL$ it can be shown that

$$\frac{\partial VaR_{\alpha}(w)}{\partial w_i} = E[R_{it} \mid R_{p,t} = VaR_{\alpha}(w)], \quad i = 1, \ldots, n$$

$$\frac{\partial ETL_{\alpha}(w)}{\partial w_i} = E[R_{it} \mid R_{p,t} \leq VaR_{\alpha}(w)], \quad i = 1, \ldots, n$$

Note: Analytic results are available under normality
Semi-Parametric Estimation

Factor Model Monte Carlo semi-parametric estimates

\[
\hat{E}[R_{it} \mid R_{p,t} = \text{VaR}_\alpha (w)] = \frac{1}{m} \sum_{t=1}^{B} R_{it}^* \cdot 1\{\text{VaR}_\alpha (w) - \epsilon \leq R_{p,t}^* \leq \text{VaR}_\alpha (w) + \epsilon\}
\]

\[
\hat{E}[R_{it} \mid R_t \leq \text{VaR}_\alpha (w)] = \frac{1}{[B\alpha]} \sum_{t=1}^{B} R_{it}^* \cdot 1\{R_{p,t}^* \leq \text{VaR}_\alpha (w)\}
\]

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FoHF Portfolio Returns and 5% VaR Violations

Constituant fund returns when FoHF returns <= 5% VaR

Fund marginal contribution to portfolio 5% ETL
Example FoHF Portfolio Analysis

- Equally weighted portfolio of 12 large hedge funds
- Strategy disciplines: 3 long-short equity (LS-E), 3 event driven multi-strat (EV-MS), 3 direction trading (DT), 3 relative value (RV)
- Factor universe: 52 potential risk factors
- $R^2$ of factor model for portfolio $\approx 75\%$, average $R^2$ of factor models for individual hedge funds $\approx 45\%$

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FMMC FoHF Returns

\[ \sigma_{FM} = 1.42\% \]
\[ \sigma_{FM,EWMA} = 1.52\% \]
\[ \text{VaR}_{0.0167} = -3.25\% \]
\[ \text{ETL}_{0.0167} = -4.62\% \]

50,000 simulations
## Factor Risk Contributions

### Factor Group % Contribution to Risk

<table>
<thead>
<tr>
<th>Factor Group</th>
<th>EWMA Std Dev</th>
<th>Expected Tail Loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity</td>
<td>46.3%</td>
<td>54.8%</td>
</tr>
<tr>
<td>Rates</td>
<td>-1.8%</td>
<td>-2.0%</td>
</tr>
<tr>
<td>Credit</td>
<td>16.3%</td>
<td>18.6%</td>
</tr>
<tr>
<td>Currency</td>
<td>-3.4%</td>
<td>-2.0%</td>
</tr>
<tr>
<td>Commodity</td>
<td>4.8%</td>
<td>5.6%</td>
</tr>
<tr>
<td>Strategy - Observable</td>
<td>-0.5%</td>
<td>-5.1%</td>
</tr>
<tr>
<td>Strategy - PCA</td>
<td>12.8%</td>
<td>22.7%</td>
</tr>
<tr>
<td>Specific</td>
<td>25.5%</td>
<td>7.3%</td>
</tr>
</tbody>
</table>

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Hedge Fund Risk Contributions

Top 8 Programs by % Expected Tail Loss

- **EWMA standard deviation**
- **Standard deviation**
- **Expected Tail Loss**

<table>
<thead>
<tr>
<th>Hedge Fund</th>
<th>Capital Allocation</th>
<th>EWMA SD Contribution</th>
<th>SD Contribution</th>
<th>ETL Contribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fund 3: LS-E</td>
<td>8.3%</td>
<td>23.5%</td>
<td>25.4%</td>
<td>24.9%</td>
</tr>
<tr>
<td>Fund 1: LS-E</td>
<td>8.3%</td>
<td>17.1%</td>
<td>15.7%</td>
<td>17.1%</td>
</tr>
<tr>
<td>Fund 4: EV-MS</td>
<td>8.3%</td>
<td>12.1%</td>
<td>9.0%</td>
<td>15.1%</td>
</tr>
<tr>
<td>Fund 9: DT-M</td>
<td>8.3%</td>
<td>8.5%</td>
<td>9.2%</td>
<td>8.4%</td>
</tr>
<tr>
<td>Fund 5: EV-MS</td>
<td>8.3%</td>
<td>6.3%</td>
<td>5.5%</td>
<td>8.1%</td>
</tr>
<tr>
<td>Fund 7: DT-M</td>
<td>8.3%</td>
<td>3.3%</td>
<td>6.4%</td>
<td>6.5%</td>
</tr>
<tr>
<td>Fund 6: EV-MS</td>
<td>8.3%</td>
<td>12.4%</td>
<td>13.3%</td>
<td>6.3%</td>
</tr>
<tr>
<td>Fund 12: RV-MS</td>
<td>8.3%</td>
<td>4.5%</td>
<td>4.4%</td>
<td>6.0%</td>
</tr>
<tr>
<td>Other</td>
<td>33.3%</td>
<td>12.1%</td>
<td>11.1%</td>
<td>7.6%</td>
</tr>
</tbody>
</table>
## Hedge Fund Risk Contribution

### Top 5 Programs by **Lowest** Marginal EWMA Standard Deviation

<table>
<thead>
<tr>
<th>Program</th>
<th>Capital Allocation</th>
<th>Marginal Short Risk</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fund 11: RV-R</td>
<td>8.3%</td>
<td>0.5%</td>
</tr>
<tr>
<td>Fund 2: LS-E</td>
<td>8.3%</td>
<td>0.8%</td>
</tr>
<tr>
<td>Fund 7: DT-M</td>
<td>8.3%</td>
<td>1.0%</td>
</tr>
<tr>
<td>Fund 8: DT-F</td>
<td>8.3%</td>
<td>1.1%</td>
</tr>
<tr>
<td>Fund 10: RV-MS</td>
<td>8.3%</td>
<td>1.2%</td>
</tr>
</tbody>
</table>

### Top 5 Programs by **Lowest** Marginal Standard Deviation

<table>
<thead>
<tr>
<th>Program</th>
<th>Capital Allocation</th>
<th>Marginal Long Risk</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fund 2: LS-E</td>
<td>8.3%</td>
<td>0.5%</td>
</tr>
<tr>
<td>Fund 11: RV-R</td>
<td>8.3%</td>
<td>0.6%</td>
</tr>
<tr>
<td>Fund 10: RV-MS</td>
<td>8.3%</td>
<td>0.8%</td>
</tr>
<tr>
<td>Fund 8: DT-F</td>
<td>8.3%</td>
<td>1.2%</td>
</tr>
<tr>
<td>Fund 12: RV-MS</td>
<td>8.3%</td>
<td>1.2%</td>
</tr>
</tbody>
</table>

### Top 5 Programs by **Lowest** Marginal Expected Tail Loss

<table>
<thead>
<tr>
<th>Program</th>
<th>Capital Allocation</th>
<th>Maginal ETL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fund 8: DT-F</td>
<td>8.3%</td>
<td>-0.8%</td>
</tr>
<tr>
<td>Fund 2: LS-E</td>
<td>8.3%</td>
<td>1.5%</td>
</tr>
<tr>
<td>Fund 11: RV-R</td>
<td>8.3%</td>
<td>1.5%</td>
</tr>
<tr>
<td>Fund 10: RV-MS</td>
<td>8.3%</td>
<td>2.0%</td>
</tr>
<tr>
<td>Fund 12: RV-MS</td>
<td>8.3%</td>
<td>3.3%</td>
</tr>
</tbody>
</table>
Summary and Conclusions

• Factor models are widely used in academic research and industry practice and are well suited to modeling asset returns

• Tail risk measurement and management of portfolios poses unique challenges that can be overcome using Factor Model Monte Carlo methods