Financial Econometrics and Volatility Models
Stochastic Volatility

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Outline

- Stochastic Volatility and Stylized Facts for Returns
- Log-Normal Stochastic Volatility (SV) Model
- SV Model with Student-t Errors
- Asymmetric SV Model
- Multivariate SV Model

Reading

- APDVP, chapters 8 and 11
- FMUND, chapter 4 (section 7)

Stochastic Volatility and Stylized Facts for Returns

Assume daily cc returns can be described as

$$r_t = \mu + \sigma_t u_t$$

where

1. σ_t is a positive random variable s.t. $\text{var}(\sigma_t | r_{t-1}, r_{t-2}, \ldots) > 0$

2. $\{\sigma_t\}$ is stationary, $E[\sigma_t^4] < \infty$ and $\rho_{\tau,\sigma^2} = \text{corr}(\sigma_t^2, \sigma_{t+\tau}^2) > 0$ for all τ

3. $u_t \sim iid\ (0, 1)$

4. $\{u_t\}$ and $\{\sigma_t\}$ are independent

SV vs. ARCH

The ARCH model is expressed as

$$\begin{aligned} r_t &= \mu + \sigma_t u_t \\ \sigma_t^2 &= a_0 + a_1 r_{t-1}^2 \end{aligned}$$

However,

$$\begin{gathered} \text{var}(\sigma_t^2 | r_{t-1}, r_{t-2}, \ldots,) = E[\sigma_t^4 | r_{t-1}, r_{t-2} \ldots] - E[\sigma_t^2 | r_{t-1}, r_{t-2} \ldots]^2 \\ = E[(a_0 + a_1 r_{t-1}^2)^2 | r_{t-1}, r_{t-2} \ldots] - E[a_0 + a_1 r_{t-1}^2 | r_{t-1}, r_{t-2} \ldots]^2 \\ = (a_0 + a_1 r_{t-1}^2)^2 - \left(a_0 + a_1 r_{t-1}^2\right)^2 = 0 \end{gathered}$$

so that there is no unpredictable volatility component.

SV vs ARCH

- SV specification can be motivated by economic theory
- Discrete-time SV specification has continuous-time diffusion representation
- SV fits nicely into continuous-time finance theory

Properties of Returns in SV Model

Key result: Because $\{u_t\}$ and $\{\sigma_t\}$ are independent, for any functions f_1 and f_2 we have

$$\begin{aligned} & E[f_1(\sigma_t, \sigma_{t-1}, \ldots,)f_2(u_t, u_{t-1}, \ldots,)] \\ = \; & E[f_1(\sigma_t, \sigma_{t-1}, \ldots,)]E[f_2(u_t, u_{t-1}, \ldots,)] \end{aligned}$$

Moments

$$\begin{aligned} E[r_t - \mu] &= E[\sigma_t u_t] = E[\sigma_t]E[u_t] = 0 \\ \text{var}(r_t) &= E[(r_t - \mu)^2] = E[\sigma_t^2 u_t^2] = E[\sigma_t^2]E[u_t^2] = E[\sigma_t^2] \end{aligned}$$

Moments continued

$$
\begin{aligned}
E[(r_t-\mu)^4] &= E[\sigma_t^4 u_t^4] = E[\sigma_t^4]E[u_t^4] = \text{kurt}(u_t)E[\sigma_t^4] \\
\text{kurt}(r_t) &= \frac{E[(r_t-\mu)^4]}{E[\sigma_t^2]^2} = \frac{\text{kurt}(u_t)E[\sigma_t^4]}{E[\sigma_t^2]^2} \\
&= \text{kurt}(u_t)\left(1+\frac{\text{var}(\sigma_t^2)}{E[\sigma_t^2]^2}\right) > \text{kurt}(u_t)
\end{aligned}
$$

Autocovariances and Autocorrelations

$$
\begin{aligned}
\gamma_{\tau,r} &= \text{cov}(r_t, r_{t+\tau}) = \text{cov}(\sigma_t u_t, \sigma_{t+\tau}u_{t+\tau}) \\
&= E[\sigma_t u_t \sigma_{t+\tau} u_{t+\tau}] - E[\sigma_t u_t]E[\sigma_{t+\tau}u_{t+\tau}] \\
&= E[\sigma_t \sigma_{t+\tau}]E[u_t u_{t+\tau}] - E[\sigma_t]E[u_t]E[\sigma_{t+\tau}]E[u_{t+\tau}] = 0
\end{aligned}
$$

Define $s_t = (r_t - \mu)^2 = \sigma_t^2 u_t^2$. Then

$$\begin{aligned}
\gamma_{\tau,s} &= \mathrm{cov}(s_t, s_{t+\tau}) = \mathrm{cov}(\sigma_t^2 u_t^2, \sigma_{t+\tau}^2 u_{t+\tau}^2) \\
&= E[\sigma_t^2 u_t^2 \sigma_{t+\tau}^2 u_{t+\tau}^2] - E[\sigma_t^2 u_t^2] E[\sigma_{t+\tau}^2 u_{t+\tau}^2] \\
&= E[\sigma_t^2 \sigma_{t+\tau}^2] E[u_t^2] E[u_{t+\tau}^2] - E[\sigma_t^2] E[u_t^2] E[\sigma_{t+\tau}^2] E[u_{t+\tau}^2] \\
&= E[\sigma_t^2 \sigma_{t+\tau}^2] - E[\sigma_t^2] E[\sigma_{t+\tau}^2] \\
&= \mathrm{cov}(\sigma_t^2, \sigma_{t+\tau}^2) = \gamma_{\tau,\sigma^2} > 0
\end{aligned}$$

Positive dependence in squared returns result from positive dependence in σ_t^2

Note:

$$\rho_{\tau,s} = \frac{\mathrm{cov}(s_t, s_{t+\tau})}{\mathrm{var}(s_t)} = \frac{\mathrm{cov}(\sigma_t, \sigma_{t+\tau})}{\mathrm{var}(\sigma_t^2)} \frac{\mathrm{var}(\sigma_t^2)}{\mathrm{var}(s_t)} = \rho_{\tau,\sigma^2} \left[\frac{\mathrm{var}(\sigma_t^2)}{\mathrm{var}(s_t)} \right]$$

Define

$$a_t = |r_t - \mu| = \sigma_t |u_t|$$

Then for $\tau > 0$

$$\begin{aligned} E[a_t^p a_{t+\tau}^p] &= E[\sigma_t^p \sigma_{t+\tau}^p |u_t|^p |u_{t+\tau}|^p] \\ &= E[\sigma_t^p \sigma_{t+\tau}^p] E[|u_t|^p]^2 \\ E[a_t^p] E[a_{t+\tau}^p] &= E[\sigma_t^p] E[\sigma_{t+\tau}^p] E[|u_t|^p]^2 \end{aligned}$$

and

$$\begin{aligned} \text{cov}(a_t^p, a_{t+\tau}^p) &= E[a_t^p a_{t+\tau}^p] - E[a_t^p] E[a_{t+\tau}^p] \\ &= \left(E[\sigma_t^p \sigma_{t+\tau}^p] - E[\sigma_t^p] E[\sigma_{t+\tau}^p] \right) E[|u_t|^p]^2 \\ &= \gamma_{\tau, \sigma^p} E[|u_t|^p]^2 \text{ for } \tau > 0 \end{aligned}$$

Note:

$$\begin{aligned}\text{var}(a_t^p) &= E[a_t^{2p}] - E[a_t^p]^2 \\ &= E[\sigma_t^{2p}|u_t|^{2p}] - E[\sigma_t^p|u_t|^p]^2 \\ &= E[\sigma_t^{2p}]E[|u_t|^{2p}] - E[\sigma_t^p]^2E[|u_t|^p]^2\end{aligned}$$

Autocorrelations of $|a_t|^p$

Define

$$A(p) = \frac{E[\sigma_t^{2p}]}{E[\sigma_t^p]^2} \text{ and } B(p) = \frac{E[|u_t|^{2p}]}{E[|u_t|^p]^2}$$

Taylor (1994) derived the following result

$$\begin{aligned}
\rho_{\tau,a^p} &= \text{corr}(a_t^p, a_{t+\tau}^p) = \frac{\text{cov}(a_t^p, a_{t+\tau}^p)}{\text{var}(a_t^p)} \\
&= \frac{\gamma_{\tau,\sigma^p} E[|u_t|^p]^2}{E[\sigma_t^{2p}]E[|u_t|^{2p}] - E[\sigma_t^p]^2 E[|u_t|^p]^2} = C(p)\rho_{\tau,\sigma^p} \\
C(p) &= \frac{A(p) - 1}{A(p)B(p) - 1} \leq \frac{1}{B(p)}
\end{aligned}$$

Result: If $u_t \sim iid\ N(0,1)$ then

$$
\begin{aligned}
E[|u_t|^p] &= 2^{p/2}\pi^{-1/2}\Gamma((p+1)/2) \\
\Gamma(u) &= \int_0^\infty x^{u-1}e^{-x}dx,\ u>0 \\
\Gamma(1/2) &= \sqrt{\pi},\ \Gamma(1)=1,\ \Gamma(u+1)=u\Gamma(u) \\
\Gamma(n) &= (n-1)! \text{ if } n \text{ is an integer}
\end{aligned}
$$

If $u_t \sim iid$ Student's t with v df then

$$
E[|u_t|] = \frac{2\sqrt{v-2}\,\Gamma((v+1)/2)}{\sqrt{\pi}(v-1)\Gamma(v/2)}
$$

The Log-Normal AR(1) Stochastic Volatility Model

$$r_t = \mu + \sigma_t u_t$$

$$\ln(\sigma_t) - \alpha = \phi(\ln(\sigma_{t-1} - \alpha) + \eta_t, |\phi| < 1$$

$$\begin{pmatrix} u_t \\ \eta_t \end{pmatrix} \sim iid\ N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \sigma_\eta^2 \end{pmatrix}\right)$$

Note

$$\ln(\sigma_t) \sim N(\alpha, \beta^2)$$

$$\beta^2 = \frac{\sigma_\eta^2}{1-\phi^2} \Rightarrow \sigma_\eta^2 = \beta^2(1-\phi^2)$$

Log Normal Distribution

Definition: If $\ln(Y) \sim N(\mu, \sigma^2)$ then $Y \sim LN(\mu, \sigma^2)$ such that

$$
\begin{aligned}
f(y|\mu, \sigma^2) &= \frac{1}{y\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln(y) - \mu}{\sigma}\right)^2\right), \; y > 0 \\
E[Y^n] &= \exp\left(n\mu + \frac{1}{2}n^2\sigma^2\right) \\
E[Y] &= \exp\left(\mu + \frac{1}{2}\sigma^2\right) \\
\text{var}(Y) &= \exp\left(2\mu + \sigma^2\right)\left(\exp(\sigma^2) - 1\right)
\end{aligned}
$$

In the Log-Normal AR(1) SV model

$$\sigma_t \sim LN(\alpha, \beta^2)$$

Alternative Parameterization

Some authors specify the log-Normal SV model as

$$\begin{aligned} r_t &= \mu + \exp(w_t/2)u_t \\ w_t - \alpha_w &= \phi(w_{t-1} - \alpha_w) + \eta_{w,t},\ \eta_{w,t} \sim iid\ N(0, \sigma^2_{\eta_w}) \end{aligned}$$

Here

$$w_t = \ln(\sigma_t^2) = 2\ln(\sigma_t) \text{ and } \sigma_t = \exp(w_t/2)$$

It follows that

$$\begin{aligned} \alpha_w &= E[w_t] = 2E[\ln(\sigma_t)] = 2\alpha \\ \beta_w^2 &= \text{var}(w_t) = \text{var}(2\ln(\sigma_t)) = 4\text{var}(\ln(\sigma_t)) = 4\beta^2 \\ \sigma^2_{\eta_w} &= \beta_w^2(1 - \phi^2) = 4\beta^2(1 - \phi^2) \end{aligned}$$

Basic Properties

- $\{r_t\}$ is strictly stationary
- All moments of r_t are finite
- $\text{kurt}(r_t) = 3\exp(4\beta^2)$
- $\text{cov}(r_t, r_{t+\tau}) = 0$ ($\{r_t - \mu, I_t\}$ is a MDS)
- $\text{cov}(s_t, s_{t+\tau}) > 0$ when $\phi > 0$, $s_t = (r_t - \mu)^2$
- ACF of $a_t^p = |r_t - \mu|^p$ behaves like ACF of s_t

Extensions of Standard SV Model

- Fat tailed distribution for u_t (e.g. Student's t)

- Dependence between u_t and η_t to capture leverage effect

- Long memory behavior for $\ln(\sigma_t)$

- Multivariate formulation

Density and Moments

$$\begin{aligned} r_t - \mu &= \sigma_t u_t = \text{log-Normal} \times \text{Normal} \\ &\Rightarrow \text{no closed form expression for density} \end{aligned}$$

Derivation of Moments

Exploit independence between $\{\sigma_t\}$ and $\{u_t\}$

Utilize moments of log-Normal distribution

Absolute Moments

$$E[|r_t - \mu|^p] = E[a_t^p] = E[\sigma_t^p u_t^p] = E[\sigma_t^p]E[|u_t|^p]$$

Now

$$\begin{aligned} \ln(\sigma_t^p) &= p\ln(\sigma_t) \sim N(p\mu, p^2\beta^2) \\ &\Rightarrow \sigma_t^p \sim LN(p\mu, p^2\beta^2) \end{aligned}$$

Hence

$$E[\sigma_t^p] = \exp\left(p\alpha + \frac{1}{2}p^2\beta^2\right)$$

Furthermore, recall for $u_t \sim iid\, N(0,1)$

$$E[|u_t|^p] = 2^{p/2}\pi^{-1/2}\Gamma((p+1)/2)$$

Therefore,

$$E[|r_t - \mu|^p] = E[\sigma_t^p]E[|u_t|^p] = \exp\left(p\alpha + \frac{1}{2}p^2\beta^2\right)2^{p/2}\pi^{-1/2}\Gamma\left(\frac{p+1}{2}\right)$$

For $p = 1$ and $p = 2$

$$
\begin{aligned}
E[|r_t - \mu|] &= \exp\left(\alpha + \frac{1}{2}\beta^2\right)\sqrt{2/\pi} \\
E[|r_t - \mu|^2] &= \text{var}(r_t) = \exp\left(2\alpha + 2\beta^2\right)\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}\right) \\
&= \exp\left(2\alpha + 2\beta^2\right)
\end{aligned}
$$

because

$$
\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2} + 1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}
$$

Straightforward algebra gives

$$
\text{kurt}(r_t) = \frac{E[|r_t - \mu|^4]}{\text{var}(r_t)^2} = 3\exp\left(4\beta^2\right)
$$

Autocorrelations

- $\{r_t - \mu\} = \{\sigma_t u_t\}$ is a MDS $\Rightarrow \{r_t\}$ is an uncorrelated processes

- $\{\sigma_t\}$ is autocorrelated because $\ln(\sigma_t)$ follows an AR(1) process

- $a_t = |r_t - \mu| = \sigma_t |u_t|$, $l_t = \ln a_t = \ln(\sigma_t) + \ln(|u_t|)$ and $s_t = (r_t - \mu)^2$ are autocorrelated and behave similarly to $\{\sigma_t\}$

Autocorrelations of l_t, σ_t, a_t, and s_t

Autocorrelations of $l_t = \ln(a_t) = \ln(\sigma_t) + \ln(|u_t|)$

$$\begin{aligned}\text{cov}(l_t, l_{t+\tau}) &= \text{cov}(\ln(\sigma_t) + \ln(|u_t|), \ln(\sigma_{t+\tau}) + \ln(|u_{t+\tau}|)) \\ &= \text{cov}(\ln(\sigma_t), \ln(\sigma_{t+\tau})) \quad \text{(b/c } u_t \text{ is iid)} \\ &= \phi^\tau \beta^2 \quad \text{(b/c } \ln(\sigma_t) \text{ follows an AR(1))}\end{aligned}$$

Then

$$\rho_{\tau,l} = \text{corr}(l_t, l_{t+\tau}) = \frac{\text{cov}(l_t, l_{t+\tau})}{\text{var}(l_t)} = \frac{\phi^\tau \beta^2}{\text{var}(\ln(\sigma_t) + \ln(|u_t|))}$$

Now

$$\text{var}(\ln(\sigma_t)) = \beta^2,\ \text{var}(\ln(|u_t|)) = \pi^2/8$$

Hence

$$\begin{aligned}\rho_{\tau,l} &= \frac{\phi^\tau \beta^2}{\beta^2 + \pi^2/8} = \frac{8\phi^\tau \beta^2}{8\beta^2 + \pi^2} = C(0, \beta) \\ \text{sign}(\rho_{\tau,l}) &= \text{sign}(\phi)\end{aligned}$$

Autocorrelations of σ_t^p

As $\ln(\sigma_t^p)$ is a Gaussian AR(1) process, with mean αp, variance $p^2\beta^2$ and autoregressive coefficient ϕ it can be shown that

$$\ln(\sigma_t^p) + \ln(\sigma_{t+\tau}^p) = \ln(\sigma_t^p \sigma_{t+\tau}^p) \sim N(2p\alpha, 2(1+\phi^{|\tau|})p^2\beta^2)$$

This follows since

$$\begin{aligned} E[\ln(\sigma_t^p) + \ln(\sigma_{t+\tau}^p)] &= p\alpha + p\alpha = 2p\alpha \\ \text{var}(\ln(\sigma_t^p) + \ln(\sigma_{t+\tau}^p)) &= \text{var}(\ln(\sigma_t^p)) + \text{var}(\ln(\sigma_{t+\tau}^p)) \\ &\quad + 2\text{cov}(\ln(\sigma_t^p), \ln(\sigma_{t+\tau}^p)) \\ &= p^2\beta^2 + p^2\beta^2 + 2p^2\beta^2\phi^{|\tau|} \\ &= 2(1+\phi^{|\tau|})p^2\beta^2 \end{aligned}$$

Hence

$$\sigma_t^p \sigma_{t+\tau}^p \sim LN(2p\alpha, 2(1+\phi^{|\tau|})p^2\beta^2))$$

It follows that

$$\begin{aligned} E[\sigma_t^p \sigma_{t+\tau}^p] &= \exp\left(2p\alpha + (1+\phi^\tau)p^2\beta^2\right) \\ \rho_{\tau,\sigma^p} &= \frac{\exp(p^2\beta^2\phi^\tau) - 1}{\exp(p^2\beta^2) - 1} \end{aligned}$$

Autocorrelations of $a_t^p = |r_t - \mu|^p$

Previous we stated that

$$\begin{aligned} \rho_{\tau,a^p} &= \text{corr}(a_t^p, a_{t+\tau}^p) = C(p)\rho_{\tau,\sigma^p} \\ C(p) &= \frac{A(p)-1}{A(p)B(p)-1}, A(p) = \frac{E[\sigma_t^{2p}]}{E[\sigma_t^p]^2} \text{ and } B(p) = \frac{E[|u_t|^{2p}]}{E[|u_t|^p]^2} \end{aligned}$$

Hence, it can be shown that

$$\rho_{\tau,a^p} = \frac{\exp(p^2\beta^2\phi^\tau) - 1}{B(p)\exp(p^2\beta^2) - 1}$$

When $p = 2$, we have

$$\rho_{\tau,s} = \frac{\exp(4\beta^2\phi^\tau) - 1}{3\exp(4\beta^2) - 1}$$

Log-Normal AR(1) SV Model with Student-t Errors

$$r_t = \mu + \sigma_t u_t,\ u_t \sim iid\ St(v),\ v > 2$$
$$\ln(\sigma_t) - \alpha = \phi(\ln(\sigma_{t-1} - \alpha) + \eta_t,\ \eta_t \sim iid\ N(0, \sigma^2_\eta)$$
$$u_t \text{ is independent of } \eta_t \text{ for all } t$$

Here

$$\begin{aligned} f(u|v) &= c(v)\left[1 + \frac{u^2}{v-2}\right]^{-(v+1)/2},\ v > 2 \\ c(v) &= \frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\sqrt{\pi(v-2)}} \\ E[u] &= 0,\ \text{var}(u) = 1,\ \text{kurt}(u) = \frac{3(v-2)}{v-4} \end{aligned}$$

Note: by definition

$$
\begin{aligned}
u_t &= v_t\sqrt{w_t} \\
& \quad v_t \sim iid\ N(0,1) \\
& \quad (v-2)w_t^{-1} \sim x_v^2
\end{aligned}
$$

Then we can write

$$
\begin{aligned}
r_t - \mu &= \sigma_t u_t = \sigma_t\sqrt{w_t}v_t = \sigma_t^* v_t \\
\sigma_t^* &= \sigma_t\sqrt{w_t} \\
\ln \sigma_t^* &= \ln \sigma_t + \frac{1}{2}\ln w_t \\
&= AR(1) + WN(0, \sigma_{\ln w}^2) \\
&= ARMA(1,1)
\end{aligned}
$$

Moments

$$\begin{aligned} a_t &= |r_t - \mu| = \sigma_t |u_t| \\ & u_t \sim iid\ St(v) \end{aligned}$$

Then

$$\begin{aligned} E[a_t^p] &= E[\sigma_t^p] E[|u_t|^p] \\ &= \exp\left(p\alpha + \frac{1}{2}p^2\beta^2\right) E[|u_t|^p] < \infty \text{ for } p < v \end{aligned}$$

Example

$$\begin{aligned} E[a_t] &= \exp\left(\alpha + \frac{1}{2}\beta^2\right) \frac{2(v-2)\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{\pi}(v-1)\Gamma\left(\frac{v}{2}\right)},\ v > 1 \\ E[a_t^2] &= \exp(2\alpha + 2\beta^2),\ v > 2 \\ E[a_t^4] &= \exp\left(4\alpha + 8\beta^2\right) \frac{3(v-2)}{v-4},\ v > 4 \end{aligned}$$

Note: Moment existence depending on v causes problems for GMM estimation of v.