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Quantitative risk management allows for qualitative notions such as optimality and expected returns to be put on a quantitative footing. It complements subjective risk considerations with an objective, statistical perspective. Broadly, quantitative risk management consists of two distinct elements. The first is *measurement*, which involves quantifying the overall risk of a portfolio. The second step is *analysis*, which involves gaining insight into the sources of risk, and determining whether or not they accurately reflect the views of the portfolio manager. Risk analysis is most commonly considered in the context of a particular risk measure, leading to concepts such as mean-*variance* optimization, and a definition of beta in terms of variances and covariances. Many volatility (or variance) based analytics can be extended to cover a broad class of risk measures, allowing for new perspectives of risk to be understood using standard analytics.

The 2008 market turmoil provides fresh motivation to take as broad a view of risk as possible. Financial risk is multi-faceted and must be measured and analyzed from multiple perspectives. More generally, the lackluster performance of conventional quantitative risk management practices highlights the need for an expansive, dynamic framework that is effective in all economic climates. To this end, we unite the academic literature on financial risk with a practitioner's perspective. First, we elucidate *shortfall*, which is the expected loss in a "bad" period and is a natural complement to volatility. Shortfall is essential to a risk management paradigm that provides meaningful analysis in both calm and turbulent markets. Second, we apply industry-standard (volatility-based) analysis to shortfall using several examples. By analyzing volatility and shortfall in conjunction we arrive at insights that cannot be obtained by considering either measure on its own. Finally, in a technical Appendix, we set down the mathematical details of a framework for generalized risk analysis. This framework applies to volatility and shortfall, and it extends without modification to a diverse class of risk measures.[1]

## Risk Measures

An influential paper by Markowitz (1952) marked the beginning of quantitative risk management by proposing volatility (standard deviation) as a risk measure. Volatility continues to play a central role in finance for many reasons. First, volatility always favors diversification over concentration, and simple mean-variance optimization problems admit analytic solutions. Furthermore, robust econometric models (e.g., multi-factor risk models) have been developed to forecast volatility. Finally, volatility can be traded directly on the open market (through the VIX index in the US and related indices in other countries) and indirectly using derivatives.

If portfolio returns were normally distributed, then the mean and volatility would *fully characterize* risk. However, in many cases, portfolio returns are materially non-normal. Extreme equity returns occur far more frequently than would be predicted by a normal distribution, resulting in a heavy loss tail populated by headline disasters such as the collapse of Lehman Brothers, the Long Term Capital debacle, Black Monday, and other market disruptions. If a portfolio contains significant weight in bonds or derivatives, the risk profile can take on arbitrary shapes, so

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[1] The material in the appendix extends in perfect analogy to the class of coherent risk measures, and with some modification, to the class of convex risk measures. Further details are in Föllmer and Scheid (2004).

additional measures are indispensible.  For a non-normal portfolio, no single measure completely describes portfolio risk.  In particular, volatility is insensitive to the difference between loss and gain (a distinction that is irrelevant for the symmetrical normal distribution).

The most prominent risk measure after volatility is value at risk (VaR), which was developed at JP Morgan in the early 1990s.  VaR is commonly expressed in terms of loss ($L$) relative to the mean:

$$L = -\left( R - \mathrm{E}\left[ R \right] \right) \ .$$
(1)

Value at risk is specified by a given confidence level and investment horizon, and it measures the maximum expected loss under "ordinary" market conditions.  For instance, a portfolio loss is expected to exceed its 95% one-day VaR on average once every 20 days.  Mathematically, value at risk is defined as

$$\mathrm{VaR} = Q(L) \ ,$$
(2)

where $Q(L)$ is the specified percentile of the centered loss distribution.

In principle, value at risk complements volatility because it is a *downside* measure:  it takes account of a portfolio's losses and ignores its gains.  This contributed to the inclusion of VaR in the Basel II regulatory framework.  However, certain drawbacks of VaR are well-appreciated (The Economist, 2004).  Some of these drawbacks relate to the way in which VaR is estimated or interpreted.  Even if those are addressed, problems remain in the very definition of VaR.

VaR, unlike volatility, may encourage concentration over diversification: lowering your value at risk can lead to a more concentrated portfolio.  A schematic example shows how this can happen.[2]

## Example 1:  Value at Risk and Diversification

Consider investing in two corporate bonds that are contractually identical; the only difference between the bonds is that they are issued by different firms.  Is it better to invest in one of the bonds or to diversify with some of each? Suppose that the annual default probability of both issuers is 0.7%.  This means that each firm has just under a one-in-a-hundred chance of defaulting over the next twelve months.  Defaults are improbable events and value at risk does not raise any concern: hold only one of the two bonds and your 99% VaR (considering only default risk) is zero.  This is because VaR is a worst-case scenario for an ordinary year.  In any one of the 99 tamer years in a typical hundred, the issuer does not default and investment in either bond pays dividends.

How risky is it to put half the money in each of the two bonds?   The probability that *something* goes wrong approximately doubles[3] to 1.4% and as a result, the 99% value at risk increases to little less than half the principal.  In other words, diversification raises value at risk from zero to almost half the portfolio value.  What went wrong?  Value at risk punishes the two-bond portfolio for a greater probability of *an* adverse event.  However, it is relatively insensitive to the event's *magnitude*.  The magnitudes of the adverse events differ (materially) by a factor of two.  A single-

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[2] Examples of this appear in the academic literature; this is adapted from Föllmer and Scheid (2004), Example 4.4.1.
[3] This calculation is valid if the default events are independent, or even if they have a modest positive correlation.

bond portfolio loses its entire principal, while the two-bond portfolio loses at least half, but usually no more than half, of its principal. •

**Figure 1**
One-day shortfall and Value at Risk, at 95% confidence level, plotted versus number of options in the portfolio. The portfolio is long an ETF, and short a variable number of out-of-the-money call options.
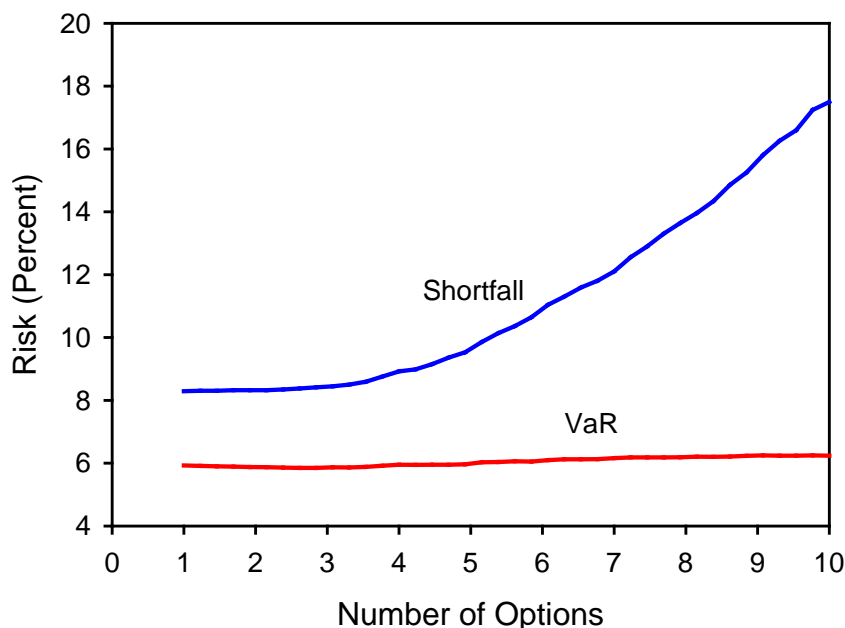


As an upper bound for loss on an ordinary day, VaR is also a lower bound for loss on an extraordinary day. *It gives no indication of what to expect on an extraordinary day*. While it is a measure of downside risk, VaR is not a measure of *extreme risk*. Are typical breaches of a VaR limit mild infractions or egregious violations? This question is beyond the scope of VaR; it can be answered only by a true measure of extreme risk.

Shortfall (*S*) is a natural extension of value at risk, and is a true extreme risk measure. It is the expected portfolio loss given that VaR has been breached:

$$S = \mathrm{E}\left[L \mid L > \mathrm{VaR}\right] \ . \tag{3}$$

In other words, shortfall is an estimate of what to expect in a "bad" period. It is impossible to hide tail risk from shortfall the way it can be hidden from VaR. In Example 1, shortfall favors the two-bond portfolio over a single-bond portfolio. More generally shortfall, like volatility but unlike VaR, will always encourage diversification. We investigate the difference between VaR and shortfall using a set of portfolios that are long an ETF and short call options on the ETF.

### Example 2: The Risk of a Short Position in Call Options

Out-of-the-money call options are commonly sold to enhance portfolio returns. The option issuer receives a premium when the option is written. Under ordinary market conditions, the underlying security experiences, at best, a modest gain, and the option expires out of the money. However, an extreme gain to the underlying can trigger a devastating loss to the issuer of the option. Using an empirically realistic (non-normal) distribution for the portfolio, we show that shortfall is a better gauge of the risk of the short position than VaR.

Consider a family of portfolios, each composed of a single share of an ETF and a short position in a variable number of identical European call options. The underlying chosen is the MSCI US Broad Market index,[4] and the analysis date is October 23, 2008. The spot price of the index is taken as $60, and the strike price of each option is $64, with one day to expiration.

If the short position consists of more than one option, the portfolio has an infinite down side – the potential to lose an arbitrarily large amount of money – since there is no upper limit to the value of the ETF. What is the one-day risk of holding such a portfolio? The risk of this family of portfolios as viewed through the lens of VaR and shortfall is illustrated in Figure 1. As the plot clearly shows, VaR does not significantly increase with the number of call options sold, while shortfall increases roughly linearly with the number of short positions.

The portfolio loses value from two types of scenarios:

        (a) **the ETF experiences a moderate loss,** in excess of the call premiums, and

        (b) **the ETF experiences a large gain,** so the call options are exercised.

Losses of the first kind are largely insensitive to the number of calls sold.[5] By contrast, losses resulting from the calls being exercised increase in direct proportion to the number of calls written.

If the call option is sufficiently deep out of the money, i.e., the strike price is sufficiently higher than the spot price, most of the losses are from the ETF losing value. In particular, the smallest of the large losses are of this kind. This implies that for such a portfolio, VaR would be quite insensitive to the number of short positions. More importantly, it disregards the impact of large losses resulting from scenarios (b). Shortfall, on the other hand, averages over all large losses including scenarios of the second kind, for which a portfolio short 10 call options will lose 10 times as much as a portfolio short a single call option. •

### Risk Analysis

Once risk has been measured, the next step is to understand the sources of risk and how they interact. Different investors may be interested in different types of sources, such as individual securities, asset classes, sectors, industries, currencies, or style factors from a particular risk model. Portfolio risk can be analyzed in terms of any type of source, so we make our discussion generic by considering sources without reference to their type.

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[4] We compute the distribution of one-day portfolio returns using the flexible semi-parametric model described by Goldberg *et al.* (2008) and Barbieri *et al.* (2008). The model uses more than 36 years of historical daily returns to the MSCI US to simulate price changes of the index and option. We standardize these returns by their historical volatility and then scale by the current volatility to give a consistent view of market history over the course of both calm and turbulent markets. We use the distribution of ETF prices at maturity of the option to directly compute the distribution of option payoffs and portfolio values. The Black-Scholes-Merton model is used only to compute the call premium given the terms and conditions of the contract and the implied volatility forecast of the model.
[5] Deep out-of-the-money call options are cheap and constitute a small portion of the initial portfolio value.

## Risk Sources

Fix an investment period and let $r_m$ denote the return to source $m$ over this period. Then the portfolio return over the period is given by a sum,

$$R = \sum_m x_m r_m \quad , \qquad (4)$$

where $x_m$ is the portfolio exposure to the source $m$. It is the job of the portfolio manager to determine the source exposures at the start of a period. The source returns are random variables, whose realized values are known only at the end of the investment period.

## Risk Contributions

Portfolio risk is not a weighted sum of source risks, so there is no direct analog to Equation 4 for risk. However, there is a parallel to Equation 4 in terms of *marginal contributions to risk* (*MCR*). The marginal contribution to risk of a source *m* is the approximate change in portfolio risk when increasing the source exposure by a small amount $\Delta x_m$, while keeping all other exposures fixed,

$$\Delta \sigma_P \approx MCR_m^\sigma \cdot \Delta x_m \quad . \qquad (5)$$

The contribution of a return source to portfolio risk is given precisely by the product of the source exposure and the marginal contribution to risk, as shown by Litterman (1996) in the context of volatility $\sigma$, and these risk contributions sum to the portfolio risk:

$$\sigma_P = \sum_m x_m \cdot MCR_m^\sigma \quad . \qquad (6)$$

In the Appendix, we show that Equation 6 holds for a large class of risk measures including shortfall, and it facilitates meaningful, even-handed decompositions of different risk measures.

## *X-Sigma-Rho* Risk Attribution

Additional insight can be provided by a refinement of the Litterman (1996) decomposition for volatility given in Equation 6. As discussed in Menchero and Poduri (2008), the marginal contribution to volatility can be expressed as the product of the stand-alone volatility of the source and the correlation of the source return with the portfolio,

$$MCR_m^\sigma = \sigma_m \cdot \rho\left(r_m, R\right) \quad . \qquad (7)$$

This gives rise to the *x-sigma-rho* risk attribution formula,

$$\sigma_P = \sum_m x_m \cdot \sigma_m \cdot \rho(r_m, R) \quad . \qquad (8)$$

This decomposition provides an intuitive framing of the concept of marginal contribution, as well as a deeper risk analysis. For instance, two assets with the same marginal contribution to risk may have very different volatility characteristics depending on their stand-alone volatilities $\sigma_m$ and correlations $\rho\left(r_m, R\right)$.

Equation 8 may be applied to shortfall using the shortfall-implied correlation, as shown in the Appendix. Shortfall correlation measures the likelihood of coincident extreme losses. By contrast, linear correlation measures the overall tendency of two sources to move together.[6]
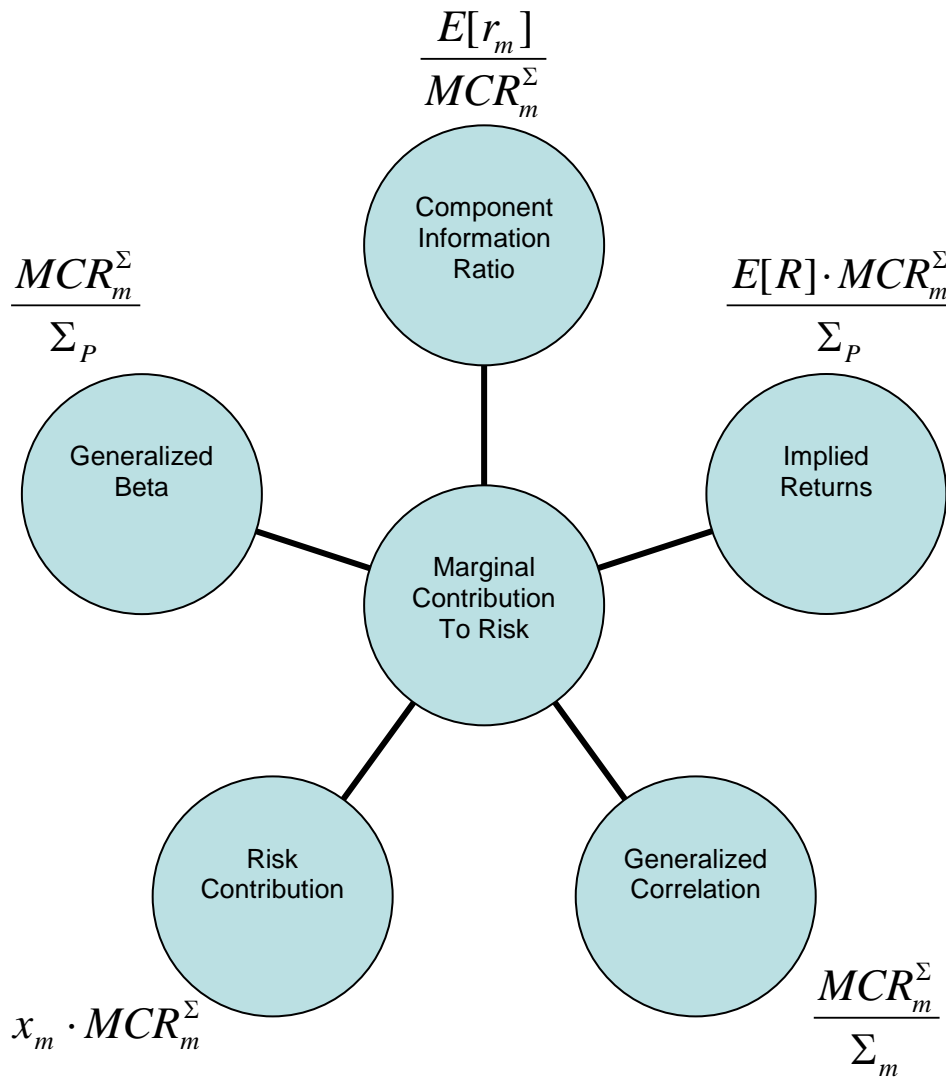
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[6] There is a conceptual parallel between shortfall correlation and exceedence correlation, as defined in Longin and Solnik (2001). Cherny and Madan (2007) provide a discussion of correlations implied by the family of coherent risk measures.

Shortfall correlation shares useful properties with linear correlation, such as scale independence, and an upper bound of 1. It also has a lower bound, which need not be equal to -1 for an asymmetric return distribution.

Examining Equations 6, 7, and 8, we see that the shared element is the marginal contribution to risk. For the special class of *linear, convex risk measures* (which includes volatility and shortfall), many additional analytics are shared through the central marginal contribution (MCR). Any one of these measures will encourage diversification, and may be decomposed using the *x-sigma-rho* formula. The relationship between marginal contribution to risk and various analytic tools is shown in Figure 2. Further details are provided in the Appendix.

**Figure 2**
Schematic diagram showing relationships between marginal contribution to risk and other widely used measures. Here, $\Sigma$ denotes a generalized risk measure.



$$\frac{E[r_m]}{MCR_m^{\Sigma}}$$

Component Information Ratio

$$\frac{MCR_m^{\Sigma}}{\Sigma_P}$$

Generalized Beta

$$\frac{E[R]\cdot MCR_m^{\Sigma}}{\Sigma_P}$$

Implied Returns

Marginal Contribution To Risk

Risk Contribution

Generalized Correlation

$$x_m \cdot MCR_m^{\Sigma}$$

$$\frac{MCR_m^{\Sigma}}{\Sigma_m}$$

We illustrate the insights provided by parallel *x-sigma-rho* decompositions of volatility and shortfall in two schematic examples. In both cases, the added perspective relies on estimates that reflect the non-normality of portfolio return distributions.

### Example 3: Portfolio Insurance

Out-of-the-money put options are commonly used to insure a portfolio against large losses. The cost of insurance is the price of the option premium. When the portfolio does not suffer a severe loss, the option expires out of the money and the premium is lost. In contrast, a severe loss to the underlying portfolio leads to a large, positive option payoff. Therefore, the value of portfolio insurance depends on the likelihood of a severe loss to the underlying portfolio. This qualitative statement can be made quantitative with a parallel analysis of volatility and shortfall.

An investor seeks to measure the reduction of risk when insuring an exchange traded fund (ETF) on the MSCI US Broad Market Index with an out-of-the-money put option on the same index. The spot price of the index is $60 and the option strike is $50. The analysis date is October 23, 2008, with the option expiration at 20 days.[7] We analyze portfolio risk in terms of both volatility and shortfall. Volatility paints an incomplete picture of this risk due to the non-normality of the return distribution of the portfolio.[8] Since a larger loss from the index generates greater option profits, the diversification benefit of holding put options increases as the risk measure becomes more concentrated in the tail of the portfolio distribution. We consider the risk of the portfolio as the option weight is varied. In Figure 3, we plot the contributions to 99% shortfall from the option and the index. Initially, the option strongly reduces the risk of the portfolio. The portfolio risk is minimized for an option weight of about 7%. Eventually, however, increasing the option weight becomes more of a gamble on a market crash, thus increasing the portfolio risk.

In Figure 4, we show the contributions of the option and the index to portfolio volatility as the option weight is varied. Qualitatively, the results are similar to Figure 3, with the portfolio risk first declining and then increasing as the option weight is varied. Note the relative amount by which the risk is reduced: the put option is much more effective at reducing 99% shortfall risk than at reducing portfolio volatility.

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[7] As in the previous example, we use the flexible semi-parametric model described by Goldberg *et al.* (2008) and Barbieri *et al.* (2008).
[8] This portfolio non-normality originates from the non-linear dependence of the option on the underlying security, as well as the non-normality of returns to the underlying security.

**Figure 3**
Contribution to 99% shortfall from index and option, as option weight is varied.  The fully hedged option weight is indicated by the vertical dashed line.  The minimum risk portfolio occurs for an option weight of about 7%.
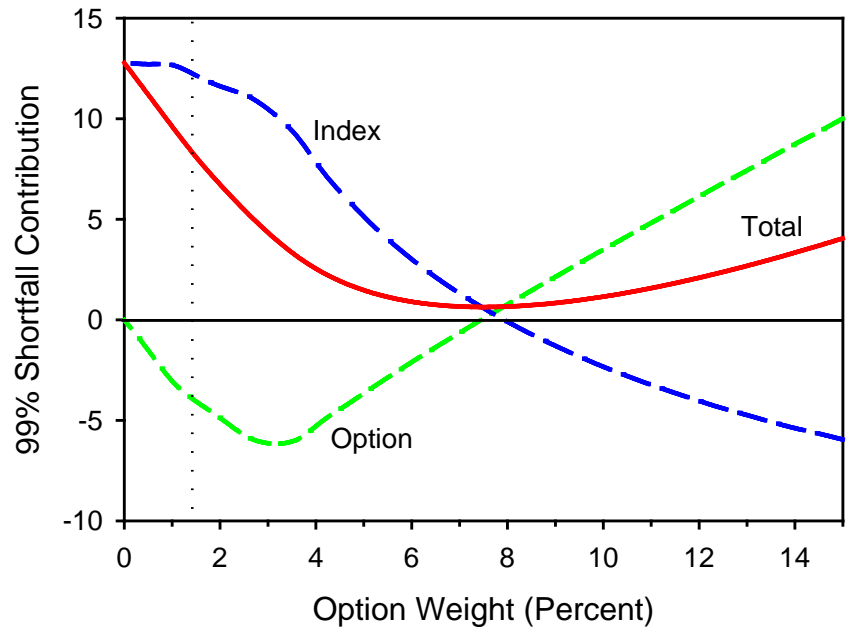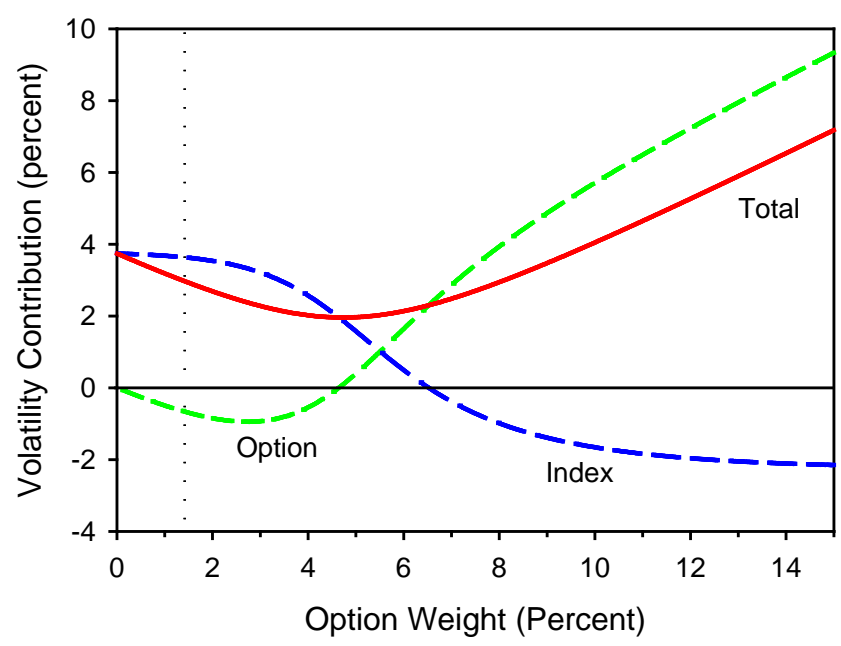


**Figure 4**
Contribution to volatility from index and option, plotted as option weights are varied.  The fully hedged option weight is indicated by the vertical dashed line.

In Table 1, we use the *x-sigma-rho* methodology to decompose three risk measures: volatility, 95% shortfall, and 99% shortfall.  The horizon in all cases is taken to be one day.  The portfolio is fully hedged, meaning that there are equal number of shares in the ETF and the option.  This results in an index weight of 98.86%, and an option weight of 1.14%, as indicated in Column 1.  In the *x-sigma-rho* framework, the risk contribution is given by the product of the exposure (*x*), the stand-alone risk (*sigma*), and the generalized correlation (*rho*).  Note that the exposures are the same for all three risk measures.

**Table 1**
Risk decomposition for volatility, 95% shortfall, and 99% shortfall, using the *x-sigma-rho* formulation.  The portfolio consists of an index and an out-of-the-money put option on the same index.

| Asset | Exposure | Stand-alone Volatility | Volatility Correlation | Volatility Contrib | 95% Stand-alone Shortfall | 95% Shortfall Correlation | 95% Shortfall Contrib | 99% Stand-alone Shortfall | 99% Shortfall Correlation | 99% Shortfall Contrib |
|---|---|---|---|---|---|---|---|---|---|---|
| Index | 98.86 | 3.74 | 0.99 | 3.67 | 8.29 | 1.00 | 8.20 | 12.78 | 1.00 | 12.63 |
| Put Option | 1.14 | 64.13 | -0.76 | -0.55 | 70.58 | -2.11 | -1.69 | 84.26 | -3.61 | -3.45 |
| Total | 100 | | | 3.12 | | | 6.51 | | | 9.18 |

The option contributes -169 basis points to 95% shortfall.  Note that the shortfall correlation is -2.11.  This value is noteworthy since linear (volatility-implied) correlation can never fall below -1.  Shortfall correlation, on the other hand, can be less than -1 due to the asymmetry of the risk measure and return distribution.  As shown in the Appendix, the shortfall correlation between the option loss $(l_m)$ and the portfolio loss $L$ is given by the ratio of two terms: (1) the expected option loss on days when the portfolio VaR is breached, and (2) the shortfall of the option.

The distribution of centered option losses is shown in Figure 5.  The right side of the distribution represents the worst 5% of option scenarios, when the 95% option VaR is breached.  The left side of the distribution represents option losses on the worst 5% of portfolio scenarios,  showing that the option performs well when the portfolio suffers losses.  The numerator of Equation A17 is the mean of the left-side of the distribution, or -148.6%.  The denominator of Equation A17 is the mean of the right side of the distribution, or 70.6%.  The correlation of -2.11 is the ratio of these two numbers. •

**Figure 5**

Partial histogram of centered option losses, defined as the option loss relative to the mean option loss $(-14.74\%)$. The theoretical maximum centered option loss is therefore 114.74%. The right side of the distribution is for bad option days, defined as days when the 95% *option* VaR (60.2%) is breached. The expected centered option loss on bad option days is 70.6%. The left side of the distribution is for days when the 95% *portfolio* VaR is breached. The mean centered option loss on bad portfolio days is -148.6% (i.e., the option performs well on bad portfolio days). The 95% shortfall correlation is the ratio of the two numbers, i.e., -2.11. Note that the left side of the distribution has a long tail, and all losses beyond -280% are trimmed for plotting purposes.
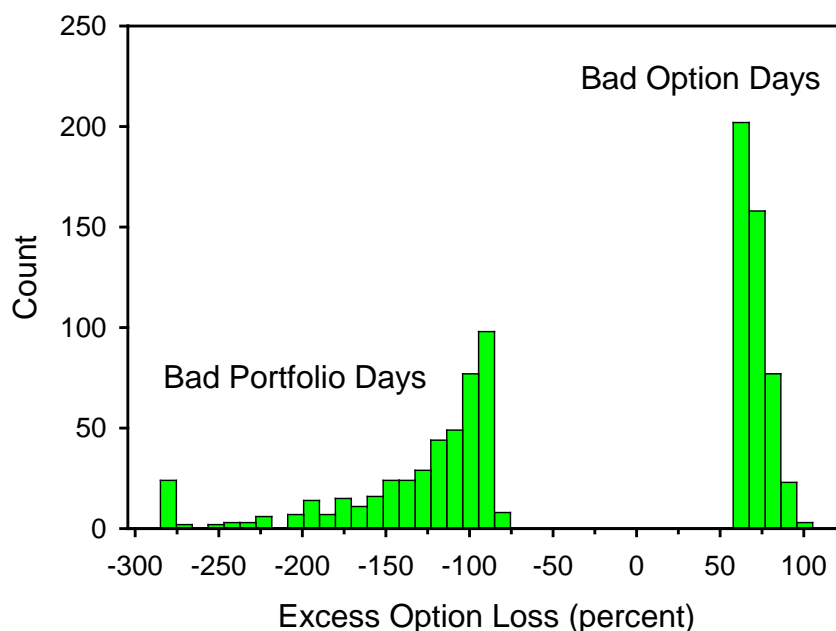


In the final example, we examine a portfolio composed of two assets that are uncorrelated by the standard volatility measure, but that nevertheless tend to experience large coincident losses. This may be the result of financial contagion, where an extreme move in one asset triggers an extreme move in another. Standard volatility and correlation measures are insensitive to the risk of extreme coincident losses. However this risk can be measured using shortfall and shortfall-implied correlation.

**Example 4: Coincident Extreme Losses**

We consider two portfolios, each composed of two equally-weighted assets whose returns follow standard normal distributions. The portfolios are distinguished only by the joint distributions of the two assets. First, we use a normal copula, which ensures that the joint distribution is also normally distributed. Copulas provide a method of formulating multivariate return distributions with given statistical properties (Nelson, 1999). The second way in which we form the joint distribution is to use the *t*-copula, which exhibits a greater likelihood of joint extreme losses. We stress that although the asset returns generated by the *t*-copula are uncorrelated using the conventional definition, they are not *independent*. Therefore, even though the asset returns are

normally distributed, the portfolio returns are fat tailed due to the increased likelihood of both assets experiencing extreme simultaneous losses.

To compute the risk of these portfolios, we take one million random draws from the two bivariate distributions. In Table 2, we perform an *x-sigma-rho* attribution of volatility and shortfall for the case of a normal copula. Since the portfolio is equally weighted, the exposure to each asset is 50%. Following the standard computations for volatility, the correlation of each asset with the portfolio is $1/\sqrt{2}$, or 0.71. The portfolio volatility is 0.71%, compared to a stand-alone volatility of 1% for each asset.

**Table 2**
*X-sigma-rho* risk decomposition for two identical assets assuming a normal copula.

| Asset | Exposure | Stand-alone Volatility | Volatility Correlation | Volatility Contrib | 95% Stand-alone Shortfall | 95% Shortfall Correlation | 95% Shortfall Contrib | 99% Stand-alone Shortfall | 99% Shortfall Correlation | 99% Shortfall Contrib |
|---|---|---|---|---|---|---|---|---|---|---|
| A | 50 | 1.00 | 0.71 | 0.35 | 2.06 | 0.71 | 0.73 | 2.67 | 0.71 | 0.94 |
| B | 50 | 1.00 | 0.71 | 0.35 | 2.06 | 0.71 | 0.73 | 2.67 | 0.71 | 0.94 |
| Total | 100 | | | 0.71 | | | 1.46 | | | 1.89 |

For 95% shortfall, the stand-alone risk of each asset is 2.06%. For normal distributions, the ratio of 95% shortfall to volatility is 2.06. The shortfall correlation is the same as the volatility correlation (i.e., 0.71). The 95% shortfall for the portfolio is 1.46, which is exactly 2.06 times the portfolio volatility. This is as expected, since the portfolio returns are normally distributed.

More interesting is the risk decomposition for the *t*-copula, given in Table 3. Note that the volatility of the portfolio is unchanged. The correlation of each asset with the portfolio is also unchanged compared to Table 2. In other words, as measured by volatility, the *t*-copula portfolio is no riskier than the joint-normal portfolio.

**Table 3**
*X-sigma-rho* risk decomposition for two identical assets assuming a *t*-copula.

| Asset | Exposure | Stand-alone Volatility | Volatility Correlation | Volatility Contrib | 95% Stand-alone Shortfall | 95% Shortfall Correlation | 95% Shortfall Contrib | 99% Stand-alone Shortfall | 99% Shortfall Correlation | 99% Shortfall Contrib |
|---|---|---|---|---|---|---|---|---|---|---|
| A | 50 | 1.00 | 0.71 | 0.35 | 2.06 | 0.77 | 0.80 | 2.67 | 0.85 | 1.13 |
| B | 50 | 1.00 | 0.71 | 0.35 | 2.06 | 0.77 | 0.80 | 2.67 | 0.85 | 1.13 |
| Total | 100 | | | 0.71 | | | 1.59 | | | 2.27 |

The shortfall measure, however, reflects the increased likelihood of extreme loss. The stand-alone shortfall of each asset is still 2.06%, since the asset returns individually are normally distributed. However, now the portfolio 95% shortfall is increased to 1.59, which is 2.24 times the portfolio volatility. That is, the portfolio returns are now fat tailed. The *x-sigma-rho* methodology cleanly captures the increased risk and attributes it to increased correlation. The 95% shortfall correlation is 0.77, and increases to 0.85 as we move deeper within the tail (i.e, 99% shortfall).

Figure 6 illustrates the effects of the joint distribution on portfolio risk. In Figure 6(a), we plot portfolio shortfall versus confidence level. As we move further into the tail, the portfolio with

returns following the *t*-copula appears riskier, due to the likelihood of coincident extreme losses. In Figure 6(b), we plot the shortfall correlation versus confidence level. For the joint normal distribution, shortfall correlation is 0.71 at all confidence levels. For the *t*-copula portfolio, however, the shortfall correlation increases with confidence level, thus indicating the increased likelihood of large coincident losses. •

**Figure 6**
Panel (a) shows the shortfall for a portfolio of two identical assets, assuming a joint normal distribution (dashed line) and a *t*-copula (solid line). Panel (b) shows the shortfall correlation under the same two distributional assumptions.
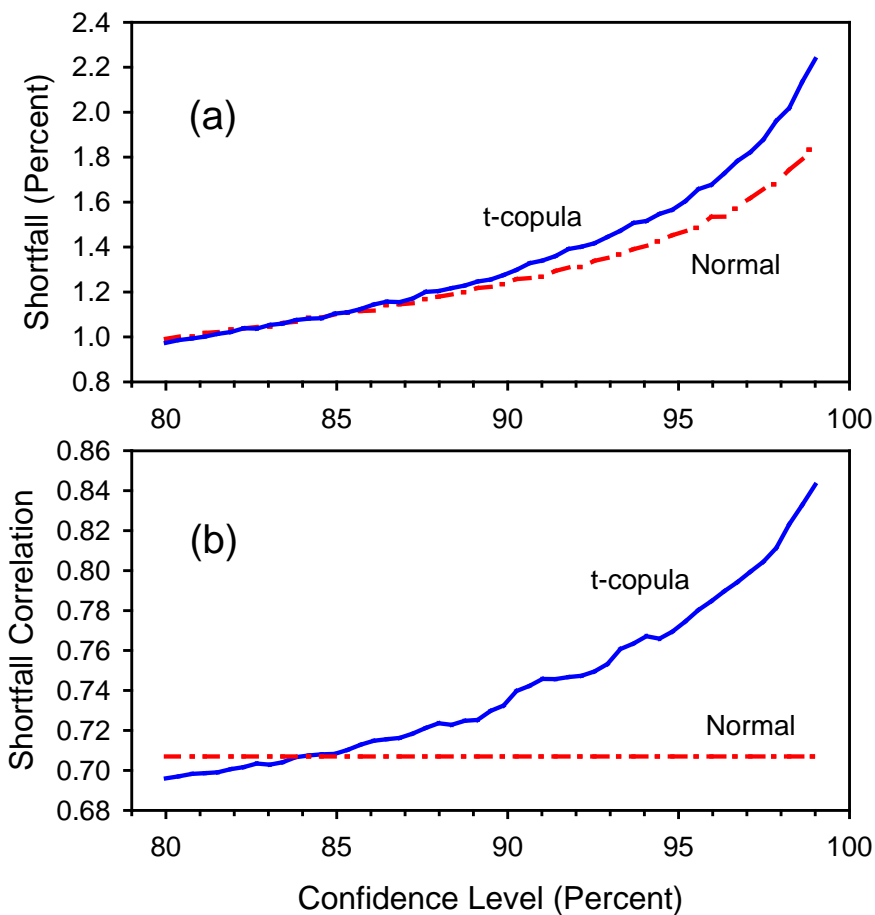
## Conclusion

The unrelenting turbulence that has plagued financial markets since 2007 highlights the importance of taking a broad, dynamic view of risk.  This means that investors need to extend standard risk management practices to include measures of extreme risk.  Shortfall, which is the expected loss given that the VaR threshold has been breached, is the most important measure of extreme risk.  It probes the tails of portfolio return distributions, promotes diversification, and is easily amenable to the tools of standard risk analysis.

Parallel decompositions of portfolio volatility and shortfall provide insights that cannot be obtained through the lens of a single risk measure.  We illustrate two such insights in this article.  The first is a deep understanding of the value of downside protection:  the diversification benefits of an out-of-the-money put are much greater for shortfall than for volatility.  The second is that shortfall correlation measures the likelihood of extreme events occurring in tandem, which is a risk that cannot be detected by linear correlation.

The key to extending industry-standard (volatility-based) risk analysis to shortfall is the marginal contribution to risk.  Its central and unifying role in risk analysis is illustrated schematically in Figure 2, and is documented mathematically in the Appendix.  The risk analysis paradigm set down there can be used as a blueprint for the evolution of risk management that will accompany the growth in our understanding of market dynamics.

## Acknowledgments

## Appendix

We define the shortfall of portfolio $P$ to be

$$S_P = \mathrm{E}\left[L \mid L > \mathrm{VaR}_P\right],$$ (A1)

where $L$ denotes the *centered* portfolio loss,

$$L = -\left(R - \mathrm{E}[R]\right),$$ (A2)

and $\mathrm{VaR}_P$ is the portfolio value at risk. It should be stressed that both value at risk and shortfall depend on investment horizon and the confidence level. For notational simplicity, however, these additional subscripts will be suppressed.

## Generalized Risk Attribution

We write the portfolio return $R$ as a sum of return contributions from various sources (e.g., assets, sectors, or factors),

$$R = \sum_m x_m r_m,$$ (A3)

where $x_m$ denotes the exposure or weight and $r_m$ denotes the return of source $m$. Since most of the discussion in this article is about loss, it is convenient to express portfolio loss as a sum of contributions from various sources (e.g., assets, sectors, or factors),

$$L = \sum_m x_m l_m,$$ (A4)

where $l_m$ denotes the centered loss of source $m$. Any risk measure $\Sigma_P$ that is scale invariant can be decomposed using Euler's theorem

$$\Sigma_P = \sum_m x_m \frac{\partial \Sigma_P}{\partial x_m}.$$ (A5)

Most familiar risk measures $\Sigma_P$, including volatility, VaR, and shortfall are scale invariant. This means that multiplying all exposures by some constant $\lambda$ also multiplies the risk by the same constant. The risk contribution of a particular source is therefore identified as

$$RC_m^{\Sigma} = x_m \cdot MCR_m^{\Sigma},$$ (A6)

where $MCR_m^{\Sigma}$ is the marginal contribution to risk of source $m$ with respect to risk measure $\Sigma$.

## Generalized Beta

The conventional definition of beta is

$$\beta_m\left(r_m, R\right) = \frac{\mathrm{cov}(r_m, R)}{\sigma_P^2},$$ (A7)

where $\sigma_P^2$ is the variance of $R$. A few lines of algebra reveal that conventional beta can also be written in terms of marginal contributions,

$$\beta_m\left(r_m, R\right) = \frac{1}{\sigma_P} \cdot MCR_m^{\sigma}.$$ (A8)

The generalized beta, for risk measure $\Sigma_P$, is a generalization of Equation A8

$$\beta_m^\Sigma \left( l_m, L \right) = \frac{1}{\Sigma_P} \cdot MCR_m^\Sigma \;.$$

(A9)

Note that in Equation A9, we have switched from returns to losses.

## Generalized Correlation
Correlation is customarily defined as

$$\rho_m \left( r_m, R \right) = \frac{\mathrm{cov}(r_m, R)}{\sigma_m \sigma_P} \;,$$

(A10)

where $\sigma_m$ is the volatility of $r_m$, and $\sigma_P$ is the volatility of $R$. Written in this form, it is not obvious how to generalize correlation for other risk measures. However, as with beta, correlation can also be expressed in terms of marginal contributions,

$$\rho_m \left( r_m, R \right) = \frac{1}{\sigma_m} \cdot MCR_m^\sigma \;.$$

(A11)

The generalized correlation, for risk measure $\Sigma_P$, is given by

$$\rho_m^\Sigma \left( l_m, L \right) = \frac{1}{\Sigma_m} \cdot MCR_m^\Sigma \;.$$

(A12)

Using Equation A6 and A12, the risk contribution of a particular source is

$$RC_m^\Sigma = x_m \cdot \Sigma_m \cdot \rho_m^\Sigma \;,$$

(A13)

and portfolio risk $\Sigma_P$ is attributed to sources according to the *x-sigma-rho* framework,

$$\Sigma_P = \sum_m x_m \cdot \Sigma_m \cdot \rho_m^\Sigma \;.$$

(A14)

That is, the portfolio risk is decomposed into the product of three terms: (1) the size of the position, (2) the stand-alone risk of the source, and (3) the generalized correlation of the portfolio and the source.

## Properties of Shortfall Correlation
The shortfall correlation between portfolio loss $L$ and component $l_m$ is given by Equation A12. Substituting Equation A4 into Equation A1, and taking partial derivatives, we obtain

$$\frac{\partial S_P}{\partial x_m} = \frac{\partial}{\partial x_m} \left( \sum_k x_k \mathrm{E}\left[ l_k \mid L > \mathrm{VaR}_P \right] \right) \;.$$

(A15)

Using the fact that the partial derivative with respect to the value at risk is zero, as discussed by Bertsimas *et al.* (2004), this expression simplifies to

$$\frac{\partial S_P}{\partial x_m} = \mathrm{E}\left[ l_m \mid L > \mathrm{VaR}_P \right] \;.$$

(A16)

Substituting Equation A16 and the definition for stand-alone shortfall into Equation A12, we obtain

$$\rho_m^S = \frac{\mathrm{E}\big[l_m \mid L > \mathrm{VaR}_P\big]}{\mathrm{E}\big[l_m \mid l_m > \mathrm{VaR}_m\big]} \quad . \tag{A17}$$

Like linear correlation, shortfall correlation is scale independent. This means that scaling one of the returns/losses by a constant leaves the correlation unchanged. In other words,

$$\frac{\mathrm{E}\big[\lambda l_m \mid L > \mathrm{VaR}_P\big]}{\mathrm{E}\big[\lambda l_m \mid \lambda l_m > \lambda \mathrm{VaR}_m\big]} = \frac{\mathrm{E}\big[l_m \mid L > \mathrm{VaR}_P\big]}{\mathrm{E}\big[l_m \mid l_m > \mathrm{VaR}_m\big]}, \tag{A18}$$

where we have used the linearity property of expectations.

Another important property of shortfall correlation pertains to the range of possible values. Standard correlation, of course, is bounded between $[-1,1]$. By contrast, the bounds of shortfall correlation are given by:

$$-\frac{S_m}{S_m^*} \le \rho_m^S \le 1 \quad . \tag{A19}$$

Here, $S_m^*$ is the shortfall of the *gain* of the stand-alone distribution. If the return distribution for source $m$ is symmetric, then $S_m = S_m^*$ and the shortfall correlation is bounded from below by $-1$, just as with standard correlation. If, however, the loss tail is different from the gain tail, then the stand-alone shortfall of the losses can exceed the stand-alone shortfall for the gains. In this case, the shortfall correlation can be less than $-1$, as with Example 3 in the main body.

## Reverse Optimization

Suppose that the objective is to maximize the ratio of expected excess return[9] to risk, $E[R]/\Sigma_P$. This quantity represents a generalized information ratio, which reduces to the conventional ratio when $\Sigma_P$ is volatility. For an unconstrained optimal portfolio, the derivative with respect to $x_m$ must equal zero,

$$\frac{\partial}{\partial x_m}\left(\frac{E[R]}{\Sigma_P}\right) = \frac{1}{\Sigma_P}\frac{\partial E[R]}{\partial x_m} - \frac{E[R]}{\Sigma_P^2}\frac{\partial \Sigma_P}{\partial x_m} = 0 \quad . \tag{A20}$$

Using Equation A3 and the definition of marginal contribution, Equation A20 can be rewritten as

$$E[r_m] = \left(\frac{E[R]}{\Sigma_P}\right) MCR_m^{\Sigma} \quad . \tag{A21}$$

This says that the expected excess return of the source is proportional to the marginal contribution to risk, with the constant of proportionality being the generalized information ratio.

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[9] When evaluating portfolios using the Sharpe ratio, it is important to use the return in excess of the risk-free rate of cash. For managers evaluated relative to a benchmark, it is important to consider the return in excess of the benchmark return (see Sharpe (2001) ).

## Generalized Component Information Ratios
The generalized information ratio can be rewritten in the following form,

$$\frac{E[R]}{\Sigma_P} = \sum_m \frac{x_m\,E[r_m]}{\Sigma_P}\left(\frac{x_m\,MCR_m^{\Sigma}}{x_m\,MCR_m^{\Sigma}}\right) . \tag{A22}$$

Grouping terms and simplifying, we find

$$\frac{E[R]}{\Sigma_P} = \sum_m RB_m^{\Sigma}\left(\frac{E[r_m]}{MCR_m^{\Sigma}}\right) . \tag{A23}$$

We identify $\left(E[r_m]/MCR_m^{\Sigma}\right)$ as the generalized component information ratio. This says that the generalized information ratio of the portfolio is given by the risk-weighted generalized component information ratios.

## REFERENCES

Barbieri, A., V. Dubikovsky, A. Gladkevich, L. R. Goldberg, and M. Y. Hayes, 2008. "Evaluating Risk Forecasts with Central Limits." Working Paper, *MSCI Barra*.

Bertsimas, D., G. J. Lauprete, and A. Samarov, 2004. "Shortfall as a Risk Measure: Properties, Optimization and Applications." *Journal of Economic Dynamics and Control*, 28: 1353-1381.

Brinson, G. P., L. Hood, and G. Beebower, 1986. "Determinants of Portfolio Performance." *Financial Analysts Journal*, vol.42, no. 4: 39-44.

Cherney, A. S. and D. B, Madan, 2007, "Coherent Measurement of Factor Risks," working paper, Moscow State University and University of Maryland.

The Economist, *Too Clever By Half*, Jan 22, 2004.

Föllmer, H. and A. Scheid, 2004. *Stochastic Finance: An Introduction in Discrete Time*, Second Edition, Walter deGruyter, Berlin.

Goldberg L. R., G. Miller and J. Weinstein, 2008. "Beyond Value at Risk: Forecasting Portfolio Loss at Multiple Horizons." *Journal of Investment Management*, vol. 6, no. 2: 1-26.

Jorion, P., 2001. *Value at Risk*, McGraw-Hill.

Litterman, R., 1996. "Hot Spots and Hedges." *Journal of Portfolio Management*: 52-75.

Longin, F. and B. Solnik, 2001, "Extreme Correlations of International Equity Markets," *Journal of Finance*, vol. LVI, no. 2:649-676.

Markowitz, H.M., 1952. "Portfolio Selection," *Journal of Finance*, vol. 7, no. 1: 77-91.

Menchero J., and V. Poduri, 2008. "Custom Factor Attribution." *Financial Analysts Journal*, vol. 64, no. 2: 81-92.

Nelson, R., 1999. *An Introduction to Copulas*, Springer-Verlag.

Sharpe W. F., 2001. "Budgeting and Monitoring the Risk of Defined Benefit Pension Funds." Working Paper, *Stanford University*.

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