Extreme Risk Management

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February 17, 2009‡

Abstract

Quantitative risk management relies on a constellation of tools that are used to analyze portfolio risk. We develop the standard toolkit, which includes betas, risk budgets and correlations, in a general, coherent, mnemonic framework centered around marginal risk contributions. We apply these tools to generate side-by-side analyses of volatility and expected shortfall, which is a measure of average portfolio excess of value-at-risk. We focus on two examples whose importance is highlighted by the current economic crisis. By examining downside protection provided by an out-of-the-money put option we show that the diversification benefit of the option is greater for a risk measure that is more highly concentrated in the tail of the distribution. By comparing two-asset portfolios that are distinguished only by the likelihood of coincident extreme events, we show that expected shortfall measures market contagion in a way that volatility cannot.

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‡We thank Angelo Barbieri, Patrick Burke-d'Orey, Vladislav Dubikovsky, Alexei Gladkevich, Julie Lefler, D.J. Orr, Jo Robbins, Peter Shepard, Jeremy Staum, and Fred Winik for their insights and support.

Electronic copy available at: http://ssrn.com/abstract=1341363

1 Introduction

At the heart of risk management is a constellation of tools that are used to analyze the risk profile of a portfolio. These tools can uncover unintended bets on common risk factors such as size or growth, illuminate the impact of exposure on emerging sectors such as renewable energy,[1] and reveal subtle opportunities for diversification. The standard toolkit includes marginal contribution to risk, beta, risk budgets and correlation, and it is generally used by financial practitioners to analyze volatility.

It is well documented in the academic literature that the identical tools can be applied to analyze a wide range of risk measures.[2] However, in the years of relative calm prior to the current credit crisis, financial practitioners had little motivation to overcome the many impediments to a more general analysis of risk. That has changed. The crisis has indelibly underscored the fact that extreme events are endemic to financial markets, and it has led to substantial interest in measuring and analyzing extreme risk.

The most direct measure of extreme risk is expected shortfall (ES),[3] which is the average loss beyond a threshold that distinguishes an everyday loss from an extreme loss. This threshold is known as value-at-risk (VaR), and it is specified by a quantile, or *confidence level*. For example, at the 99% confidence level, one-day value-at-risk provides an upper bound for portfolio loss 99 days out of 100. Note however, that 99% one-day value-at-risk is a best-case scenario on the worst day in 100, and it provides no indication of what to expect in times of severe distress. By contrast, the 99% one-day shortfall is an average, or expected value, for the worst day out of 100, and is thereby a natural guide to allocation of capital reserves.

When estimated from a distribution that reflects salient characteristics of empirical data—including a realistic proportion of extreme events, and the asymmetry between gains and losses observed in markets—shortfall provides information that complements volatility, which is a measure of average return dispersion.[4] Shortfall, like volatility, exhibits technical properties that facilitate analysis with the standard toolkit, and enable it to play a central role in the investment process. Shortfall is sub-additive, so that it encourages diversification, and *convex*; see Bertsimas et al. (2004), so that it supports tractable optimization routines.

In this article, we review the standard toolkit in the context of general risk measures. As illustrated in Figure 1, the most basic element is marginal contribution to risk: the other tools in the collection can be expressed in terms of marginal contributions. Subsequently, we provide examples of parallel analyses of volatility and expected shortfall. These examples highlight the properties of the two risk measures, and provide investment

[1] See Chia et al. (2009) for more about green investment risk.

[2] The risk measures must be scale-invariant, sub-additive, and convex; see Rockafellar et al. (2006), Rockafellar et al. (2007) and references therein. We explore the economic meaning of these requirements below.

[3] Under broad assumptions, *expected tail loss* and *conditional value-at-risk* are synonymous with expected shortfall.

[4] Some models of extreme risk make the assumption that portfolio returns follow a normal or conditionally normal distribution, a t-distribution with a given number of degrees of freedom, and other distributions in the *location-scale* family. Under this type of modeling assumption, extreme risk measures are scalar multiples of volatility and provide scant additional insight.

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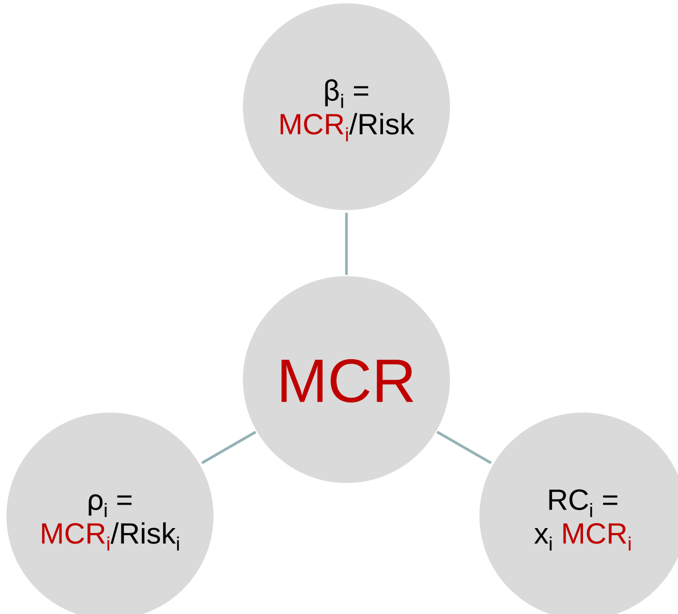

Figure 1: Relationship of the marginal contribution to risk (MCR) to the familiar concepts of beta, correlation, and risk contribution (RC); here risk may refer to volatility or expected shortfall.

insights that cannot be obtained through the lens of a single risk measure.

2 Marginal Contribution to Risk

Some basic questions that an investor might ask are:

(i) How will a given trade impact my portfolio risk?

(ii) What set of trades will most efficiently reduce my overall risk?

(iii) How should funds be allocated to asset classes, sectors or managers?

For trades that involve a sufficiently small turnover, an approximate answer to these questions is given by the *marginal contribution to risk* (MCR):

$$\mathrm{MCR}^{\mu}(P, A_i) := \frac{\partial \mu(P)}{\partial x_i}. \tag{1}$$

In Formula (1), P denotes portfolio return, $\mu(P)$ denotes portfolio risk, and x_i is the weight of portfolio component A_i. We can think of the component A_i as an asset, a sector, a linear factor, or an element of any linear decomposition of P. The marginal contribution indicates how much the portfolio risk will change upon buying or selling a

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small amount of a given asset. Because MCR is a (mathematical) derivative, it points in the direction of interest but is valid only for small changes in portfolio weights.

Marginal contribution to risk can be used to estimate implied expected return to portfolio components via a concept known as *reverse optimization* (see e.g. Sharpe (2001)). Consider a portfolio that is optimal in the sense that the ratio of return to risk is maximized. (Risk may refer to volatility, shortfall, or another measure.) Then the expected rate of return of a component A_i is proportional to its MCR.[5] Comparing the implied expected returns from reverse optimization with a manager's views may reveal unintended bets.

For a convex risk measure such as volatility or shortfall, selling a component of a portfolio decreases its MCR, and buying a component increases its MCR.[6] In terms of reverse optimization, if the expected rate of return implied by the current holdings is incompatible with a manager's views, selling a component reflects a move towards a more bearish view of the component, whereas buying a component reflects a move towards a more bullish view.

Beta

A rescaling of marginal contribution to risk leads to the concept of beta. We define the beta of a portfolio component A_i as the MCR divided by the total portfolio risk:

$$\beta^{\mu}(P, A_i) := \frac{1}{\mu(P)} \frac{\partial \mu(P)}{\partial x_i}. \tag{2}$$

It is a straightforward exercise in algebra to convert Formula (2) to the standard formula for beta, which is expressed in terms of covariances and variances, in case the risk measure μ is volatility. A comparison between Formula (2) for beta and Formula (1) for MCR shows that the beta can be interpreted as the *percent change* in risk upon buying or selling a small amount of component A_i.

Beta is also the natural measure of risk in the capital asset pricing model (CAPM). Conventionally, the CAPM is defined in terms of volatility, but it can also be defined in terms of shortfall (see e.g. Bertsimas et al. (2004)). Both the volatility and shortfall CAPMs make the assumption that the market portfolio is optimal with respect to volatility or shortfall. The CAPM is a model of the expected excess rate of return of assets, and can be viewed as a special case of reverse optimization for the market portfolio.

Risk Contribution and Risk Budget

Most standard risk measures μ including volatility, value-at-risk, and expected shortfall are scale invariant,[7] so that the risk of a portfolio with n components can be decomposed

[5]To illustrate the basis for reverse optimization, take a hypothetical two-component portfolio and suppose the expected return of component 1 is greater than the expected return of component 2. If $\text{MCR}_1 < \text{MCR}_2$, it would be possible to buy component 1 and sell component 2, increasing the overall rate of return and reducing risk. But this would violate the assumption that the portfolio is optimal.

[6]Technically, selling a component does *not increase* its MCR and vice versa.

[7]A function f is scale invariant, or positively homogeneous of degree one, if $f(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) = \lambda f(x_1, x_2, \ldots, x_n)$ for any positive λ.

using the Euler formula:

$$\mu(P) = \sum_{i=1}^{n} x_i \frac{\partial \mu(P)}{\partial x_i}$$
$$= \sum_{i=1}^{n} x_i \mathrm{MCR}^{\mu}(P, A_i).$$

This leads to the concepts of a risk contribution:

$$\mathrm{RC}^{\mu}(P, A_i) = x_i \mathrm{MCR}^{\mu}(P, A_i) \tag{3}$$

and a risk budget:

$$\mathrm{RB}^{\mu}(P, A_i) = \frac{x_i \mathrm{MCR}^{\mu}(P, A_i)}{\mu(P)}. \tag{4}$$

In an optimized portfolio, $\mathrm{MCR}^{\mu}(P, A_i)$ is proportional to expected excess return of component i, and the risk budget RB_i is the percentage of dollar-expected excess return due to portfolio component i.

Correlation

The risk contribution in Formula (3) can be further decomposed in an economically meaningful way by factoring the marginal contribution:

$$\mathrm{RC}^{\mu}(P, A_i) = x_i \mu(A_i) \frac{\mathrm{MCR}^{\mu}(P, A_i)}{\mu(A_i)}$$
$$= x_i \mu(A_i) \rho^{\mu}(P, A_i). \tag{5}$$

This is the X-Sigma-Rho decomposition described in Menchero and Poduri (2008), who show that if the risk measure μ is volatility, then the quotient

$$\rho^{\mu}(P, A_i) = \frac{\mathrm{MCR}^{\mu}(P, A_i)}{\mu(A_i)}$$

is the linear correlation between the portfolio P and the the component A_i. In case μ is expected shortfall, the correlation measures the likelihood of coincident extreme losses. We illustrate this in Section 3.2.

When the risk measure μ is convex and sub-additive, risk-implied correlation shares useful properties with linear correlation, including scale independence, an upper bound of 1, and a lower bound (that need not be -1!). These properties are discussed in the context of shortfall in the Appendix.

3 Examples

We illustrate the concepts developed above for volatility (σ), which is a measure of average loss, and shortfall (ES), which is a measure of extreme loss. We use a flexible semi-parametric model of extreme risk (Goldberg et al. (2008), Barbieri et al. (2008)), where differences between shortfall and volatility can lead to new insights into the structure of extreme risk.

3.1 Downside Protection

Simple out-of-the-money put options are commonly used to insure a portfolio against large losses. These come at the price of the option premium, which is usually not recouped. Volatility paints an incomplete picture of the risk of this type of instrument, due to the asymmetry in potential payoffs. Most of the time the payoff is small and negative, but on rare occasions it is large and positive. Volatility cannot distinguish between the seller of an out-of-the-money put and the buyer, although their respective outlook and risk tolerance differ greatly. The selling of this type of instrument has been colorfully described as "picking up pennies in front of steamrollers;" presumably the buyers are driving the steamrollers!

We examine a portfolio consisting of an exchange traded fund on a broad equity market index, and an out-of-the-money put option on the fund. For a given allocation between the fund and the option, the risk characteristics can look dramatically different through the lens of volatility compared to shortfall.

3.1.1 Methodology

Consider a portfolio consisting of an exchange traded fund (ETF) on the MSCI US Broad Market Index and an out-of-the-money put option on the index. Since a larger loss from the index generates greater option profits, the diversification benefit of holding put options increases as the risk measure becomes more concentrated in the tail of the portfolio distribution. For expected shortfall, this means that diversification benefit of the put increases with confidence level.

We use 30 years of historical daily returns to the MSCI US to simulate price changes of the index and option. We first normalize these returns by the historical volatility, and then multiply by the current volatility; this procedure gives a consistent view of market history over the course of both calm and turbulent markets. To compute the price change of the option we use the terms and conditions of the put contract,[8] the rescaled empirical index returns, and the Black-Scholes-Merton price of the option. From this large set of possible price changes, we compute the portfolio risk. This model of risk does not rely on any distributional assumptions, and is flexible enough to exactly capture the nonlinearity of the option.

3.1.2 A Fully-Hedged Portfolio

In a fully-hedged portfolio, the number of ETF shares equals the number of option contracts, resulting in portfolio weights of 99.35% for the ETF and 0.65% for the option. The total portfolio forecast one-day volatility is 3.24%, the 95% expected shortfall is 6.79%, and the 99% expected shortfall is 9.53%. The magnitude of the risk forecasts reflects the turbulent market conditions on the analysis date, 23 October 2008.

The first column in Table 1 displays the portfolio weights, which are used to compute portfolio volatility and expected shortfall at all confidence levels. The next column reports

[8] The option contract details used in the examples below are as follows: The strike price is \$50, the volatility is 50%, and the option expires in 20 days. For simplicity and without loss of generality, we ignore the risk-free rate and dividend yield of the ETF.

Table 1: Decomposition of volatility, 95% expected shortfall, and 99% expected shortfall of option and index

Asset	X	σ	ρ^{σ}	RC^{σ}	ES_{95}	$\rho^{\mathrm{ES}_{95}}$	$\mathrm{RC}^{\mathrm{ES}_{95}}$	ES_{99}	$\rho^{\mathrm{ES}_{99}}$	$\mathrm{RC}^{\mathrm{ES}_{99}}$
Index	0.9935	3.74	0.99	3.68	8.29	1	8.24	12.78	1	12.7
Put	0.0065	101.57	-0.66	-0.44	87.06	-2.55	-1.45	100.4	-4.84	-3.17
Total	1			3.24			6.79			9.53

the stand-alone volatilities of the index and the option. The stand-alone volatility of the index is 3.74%, while the option is much more volatile. The linear correlation of .99 between the ETF and the portfolio reflects the 99.35% weight of the index in the portfolio. The contribution of the index to portfolio volatility is the product $0.9935 \cdot 3.74\% \cdot 0.99 = 3.68\%$, as reported in the column labelled RC^{σ}. The diversifying effect of the put option on volatility results in a negative contribution of -0.44%. The sum of the contributions of the index (3.68%) and the option (-0.44%) is the total portfolio volatility of 3.24%.

A parallel analysis shows that the contribution of the put option to 95% shortfall is the product of the exposure of 0.65%, the stand-alone 95% expected shortfall of the option of 87.06%, and the shortfall correlation of -2.55 of option losses and portfolio losses. This results in a contribution of -1.45%. The sum of the shortfall contributions of the index and the option gives the portfolio 95% expected shortfall of 6.79%. The shortfall correlation between option loss and portfolio loss at 99% is approximately double the shortfall correlation at 95%, which indicates that the option more effectively reduces the risk of losses that are more severe. This indicates that the diversification benefits are greater as the risk measure becomes more concentrated in the tail of the portfolio loss distribution.

The value of -2.55 for the shortfall correlation is noteworthy since it is not a possible value for linear correlation, which is bounded below by -1. Shortfall correlation can be less than -1 due to the combination of the asymmetry of the risk measure with the asymmetry of the portfolio and option loss distributions. This can be seen in an explicit expression for shortfall correlation between the option O and the portfolio P:

$$\rho^{\mathrm{ES}_{\alpha}}(P, O) = \frac{\mathbf{E}[-O \mid -P > \mathrm{VaR}_{\alpha}(P)] - \mathbf{E}[-O]}{\mathbf{E}[-O \mid -O > \mathrm{VaR}_{\alpha}(O)] - \mathbf{E}[-O]}. \tag{6}$$

The financial meaning of this expression is that the shortfall correlation is the ratio of two terms: 1) the expected option loss on days when the portfolio suffers large losses, and 2) the expected option loss on days when the *option* suffers large losses (i.e., the expected shortfall of the option).[9] The centered loss distribution of the option is shown in Figure 2 on days when the portfolio value-at-risk is exceeded. (We subtract the expected mean loss of 21.73%.) The 95% value-at-risk is 76.39%, while the 95% expected shortfall (which is the denominator of Formula (6)) is 87.06%. The numerator is the average loss of the option over scenarios for which portfolio loss exceeds 95% portfolio value-at-risk.

[9]Formula (6) is discussed in greater detail in the Appendix.

A histogram of option losses for these scenarios is shown in Figure 2. Since the weight of the option is sufficiently small, index losses and portfolio losses are contemporaneous. The mean loss over these scenarios is -221.90% (a gain of 221.90%). The 95% shortfall correlation is -221.90/87.06 = -2.55.

3.1.3 Doomsday Portfolios

When the number of option contracts exceeds the number of ETF shares held in the portfolio, the hedge ratio exceeds 1, and the returns to the portfolio performs well when the index experiences extreme losses. As the weight of the option increases, the portfolio tends to become more of a gamble on a market crash.

Figure 3 shows the contribution of the option and the ETF to 99% expected shortfall of the portfolio as a function of option weight. With no option present, the shortfall contribution of the ETF is the total portfolio shortfall. When the weight of the option in the portfolio is sufficiently small, an index loss results in a portfolio loss and vice versa. Consequently, the option makes a negative contribution to portfolio shortfall. The contribution of the option to shortfall is minimal when the option weight is around 2%. At 6% option weight, the shortfall correlation between the option and the portfolio is zero.

Figure 4 shows the contribution of the option and ETF to portfolio volatility as a function of option weight. The plots for volatility and 99% expected shortfall are qualitatively similar, but the former is much less dramatic than the latter. The option contribution to volatility is minimal at an option weight of approximately 2%, and the option is linearly uncorrelated with the portfolio at a weight of approximately 3.5%.

3.2 Coincident Extreme Losses

A more subtle example of an investment not fully described by volatility is a portfolio containing two assets that are linearly uncorrelated but nevertheless tend to experience coincident large losses. The impact of coincident extreme losses can be exposed by X-Sigma-Rho attribution of shortfall.

We construct a pair of equally weighted, two-asset portfolios that differ only in the dependence of the assets. Each of the two (statistically identical) assets follows a standard normal distribution, and the dependence structures of the portfolios are given by:

N. A normal copula

T. A t-copula with 2 degrees of freedom

To compute the risk of these portfolios we take 10^6 random draws from the two bivariate distributions. Table 2 reports volatility, 95% shortfall and 99% shortfall for the two portfolios; at sufficiently large quantiles, the probability of coincident extreme losses to the assets is greater in Portfolio T than in Portfolio N. Tables 3 and 4 report X-Sigma-Rho decompositions of the three risk measures. Because the two assets are statistically identical and are equally weighted, the risk contributions are identical within each portfolio.

Figure 5 shows the shortfall of portfolios N and T at varying confidence levels. The difference in shortfall for portfolios N and T increases with confidence level, reflecting the greater likelihood of simultaneous large losses at higher quantiles.

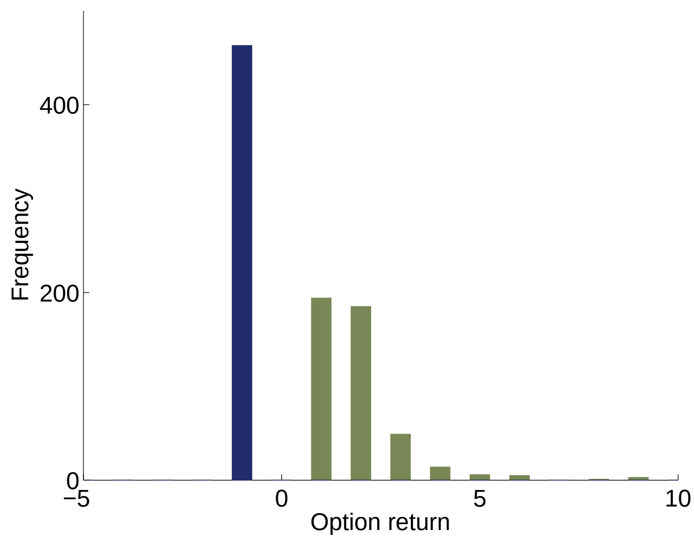

Figure 2: Option return on days when the option 95% VaR is exceeded (blue), and on days when portfolio 95% VaR is exceeded (green; the largest five gains (1000%, 1300%, 1800%, 2900%, and 6200%) are excluded for clarity).

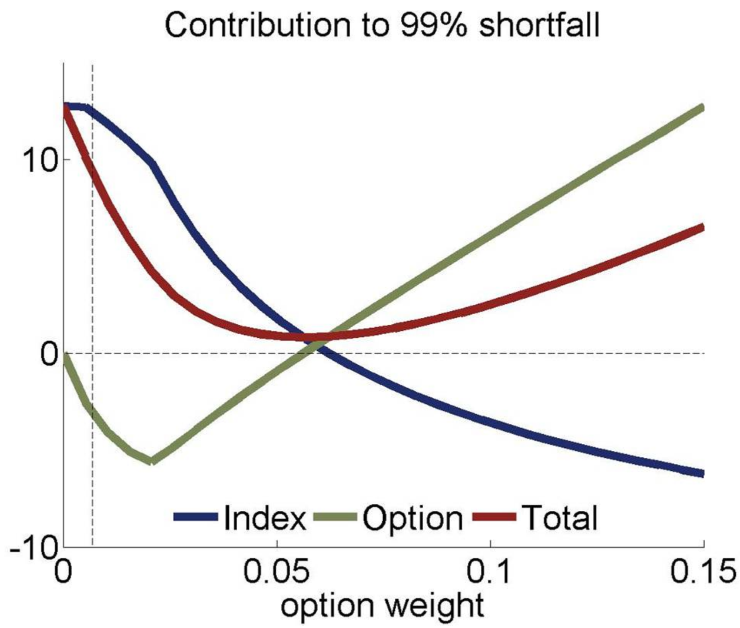

Figure 3: Contribution to 99% shortfall from index and option at varying option weights; the total weight is constrained to sum to 1, and shortfall and contributions are expressed in percentage points; the fully-hedged option weight is indicated with a vertical dashed line.

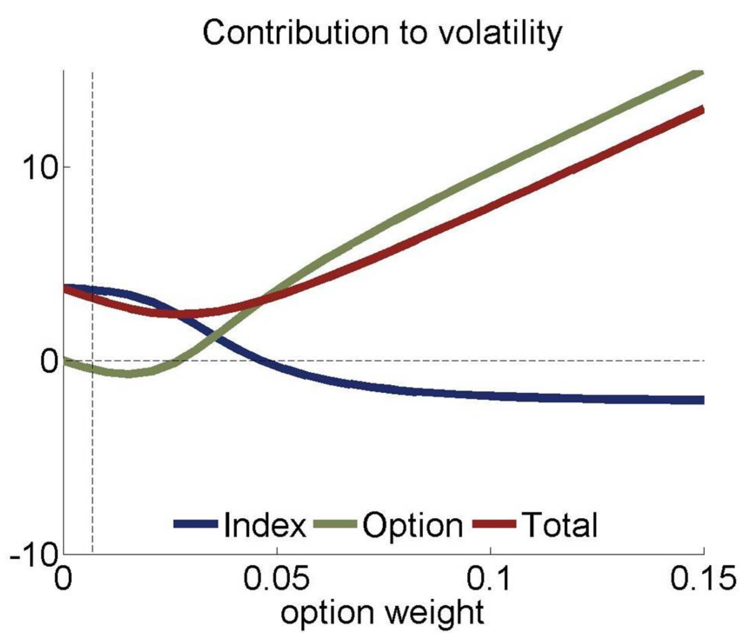

Figure 4: Contribution to volatility from index and option at varying option weights; the total weight is constrained to sum to 1, and shortfall and contributions are expressed in percentage points; the fully-hedged option weight is indicated with a vertical dashed line.

Table 2: Portfolio risk as measured by volatility, 95% shortfall, and 99% shortfall

	Volatility	95% Shortfall	99% Shortfall
N	0.71	1.46	1.89
T	0.71	1.59	2.27

Table 3: Decomposition of volatility, 95% expected shortfall, and 99% expected shortfall of two independent normally distributed assets

Asset	X	σ	ρ^{σ}	RC^{σ}	ES_{95}	$\rho^{\mathrm{ES}_{95}}$	$\mathrm{RC}^{\mathrm{ES}_{95}}$	ES_{99}	$\rho^{\mathrm{ES}_{99}}$	$\mathrm{RC}^{\mathrm{ES}_{99}}$
A	0.5	1	0.71	0.35	2.06	0.71	0.73	2.67	0.71	0.94
B	0.5	1	0.71	0.35	2.06	0.71	0.73	2.67	0.71	0.94
Total	1			0.71			1.46			1.89

The difference in shortfall for the two portfolios can be entirely explained by shortfall-implied correlation, which is reported in Tables 3 and 4 as part of the X-Sigma-Rho decomposition. For example, the 99% shortfall implied correlation between either asset and portfolio T is 0.85, which exceeds the correlation of 0.71 between the asset and Portfolio N. Figure 5 compares shortfall implied correlation between either asset and Portfolios N and T as a function of confidence level.

4 Conclusion

The global market downturn that began in 2007 highlights the importance of taking a broad view of financial risk. In this article, we demonstrate that many of the standard tools used to measure and decompose volatility can be used to analyze measures of extreme risk, such as expected shortfall. Side-by-side analyses of volatility and shortfall lead to an understanding of portfolio risk that cannot be obtained through the lens of a single risk measure. For example, by decomposing volatility and expected shortfall of a portfolio consisting of an ETF and an out-of-the-money put, we illustrate the fact that the diversification benefit of downside protection tends to be greater for a risk measure that is more highly concentrated in the tail of a portfolio. This observation may lead an investor to

Table 4: Decomposition of volatility, 95% expected shortfall, and 99% expected shortfall of two uncorrelated, normally distributed assets that exhibit extreme dependence

Asset	X	σ	ρ^{σ}	RC^{σ}	ES_{95}	$\rho^{\mathrm{ES}_{95}}$	$\mathrm{RC}^{\mathrm{ES}_{95}}$	ES_{99}	$\rho^{\mathrm{ES}_{99}}$	$\mathrm{RC}^{\mathrm{ES}_{99}}$
A	0.5	1	0.71	2.06	0.77	2.67	0.85	0.35	0.80	1.13
B	0.5	1	0.71	2.06	0.77	2.67	0.85	0.35	0.80	1.13
Total	1			0.71			1.59			2.27

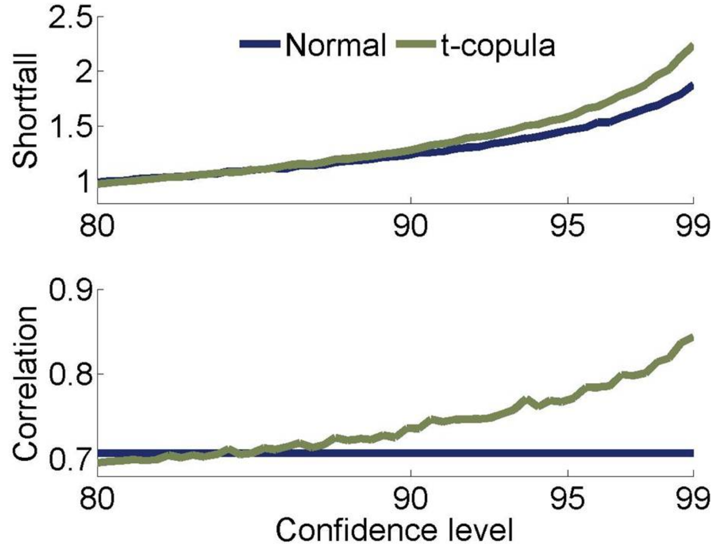

Figure 5: Shortfall and shortfall-implied correlation at varying confidence levels for portfolios N and T (normal and t-copula); portfolio N has uncorrelated extreme losses, while T exhibits correlated extreme losses.

reconsider the value of downside protection.

Volatility, expected shortfall, and many other risk measures imply correlations between a portfolio and any of its components. In general, risk-implied correlations differ from linear correlation. By examining a pair of two-asset portfolios that differ only in the likelihood that both assets suffer extreme simultaneous losses, we illustrate the fact that shortfall is sensitive to contagion risk in a way that volatility is not.

A Properties of Shortfall Correlation

We define the expected shortfall at confidence level α of portfolio P to be

$$\mathrm{ES}_\alpha(P) = \mathbf{E}[-P| - P > \mathrm{VaR}_\alpha(P)] - \mathbf{E}[-P]$$

where P denotes the portfolio return, and VaR_α is the value-at-risk at confidence level α, defined to be

$$\mathrm{VaR}_\alpha(P) = -(F^{-1}(1-\alpha) - \mathbf{E}[P])$$

where $F^{-1}(1-\alpha)$ is the $1-\alpha$ percentile of the cumulative return distribution. Note that using this definition value-at-risk is typically a positive number; expected shortfall on the other hand is always positive.

We assume that the portfolio P can be expressed as a linear combination of components, which may be assets, sectors, or factors:

$$P = \sum_{i=0}^{n} x_i A_i. \tag{7}$$

where x_i denotes exposures or weights and A_i denote asset, sectors, or factors. Note that if the A_i represent linear equity factors, Formula (7) does not hold if the portfolio contains nonlinear instruments.

Shortfall Correlation Formula. *The shortfall correlation at confidence level α between a portfolio P and a linear component A_i is given by*

$$\rho^{\text{ES}_\alpha}(P, A_i) = \frac{\mathbf{E}[-A_i \mid -P > \text{VaR}_\alpha(P)] - \mathbf{E}[-A_i]}{\mathbf{E}[-A_i \mid -A_i > \text{VaR}_\alpha(A_i)] - \mathbf{E}[-A_i]}.$$

Proof. Substituting for P using Formula (7), applying the definition of shortfall, and exploiting the linearity of conditional expectation,

$$\begin{aligned}
\frac{\partial \text{ES}_\alpha(P)}{\partial x_i} &= \frac{\partial \text{ES}_\alpha\left(-\sum_{i=1}^n x_i A_i\right)}{\partial x_i} \\
&= \frac{\partial \left(\sum_{i=1}^n x_i \left(\mathbf{E}[-A_i \mid -P > \text{VaR}_\alpha(P)] - \mathbf{E}[-A_i]\right)\right)}{\partial x_i} \\
&= \mathbf{E}[-A_i \mid -P > \text{VaR}_\alpha(P)] - \mathbf{E}[-A_i].
\end{aligned}$$

where we have used the fact that the partial derivative with respect to the VaR is zero (see Bertsimas et al. (2004)). Substituting for $\partial \text{ES}_\alpha(P)/\partial x_i$ and applying the definition of expected shortfall to $\text{ES}_\alpha(A_i)$,

$$\begin{aligned}
\rho^{\text{ES}_\alpha}(P, A_i) &= \frac{1}{\text{ES}_\alpha(A_i)} \frac{\partial \text{ES}_\alpha[P]}{\partial x_i} \\
&= \frac{\mathbf{E}[-A_i \mid -P > \text{VaR}_\alpha(P)] - \mathbf{E}[-A_i]}{\mathbf{E}[-A_i \mid -A_i > \text{VaR}_\alpha(A_i)] - \mathbf{E}[-A_i]}.
\end{aligned}$$

□

Correlation Scaling Property. *The shortfall correlation between a portfolio and one of its components is scale independent.*

Proof. The scenarios $-P > \text{VaR}_\alpha(P)$ are unchanged on multiplying P by $p > 0$. Using the shortfall correlation formula,

$$\begin{aligned}
\rho^{\text{ES}_\alpha}(pP, A_i) &= \frac{\mathbf{E}[-A_i \mid -pP > \text{VaR}_\alpha(pP)] - \mathbf{E}[-A_i]}{\mathbf{E}[-A_i \mid -A_i > \text{VaR}_\alpha(A_i)] - \mathbf{E}[-A_i]} \\
&= \frac{\mathbf{E}[-A_i \mid -P > \text{VaR}_\alpha(P)] - \mathbf{E}[-A_i]}{\mathbf{E}[-A_i \mid -A_i > \text{VaR}_\alpha(A_i)] - \mathbf{E}[-A_i]} \\
&= \rho^{\text{ES}_\alpha}(P, A_i).
\end{aligned}$$

Similarly, the scenarios $-A_i > \text{VaR}_\alpha(A_i)$ are unchanged on multiplying A_i by $a > 0$.

Exploiting this fact and the linearity of conditional expectation,

$$\begin{aligned}\rho^{\mathrm{ES}_\alpha}(P, aA_i) &= \frac{\mathbf{E}[-aA_i \mid -P > \mathrm{VaR}_\alpha(P)] - \mathbf{E}[-aA_i]}{\mathbf{E}[-aA_i \mid -aA_i > \mathrm{VaR}_\alpha(aA_i)] - \mathbf{E}[-aA_i]} \\ &= \frac{\mathbf{E}[-A_i \mid -P > \mathrm{VaR}_\alpha(P)] - \mathbf{E}[-A_i]}{\mathbf{E}[-A_i \mid -A_i > \mathrm{VaR}_\alpha(A_i)] - \mathbf{E}[-A_i]} \\ &= \rho^{\mathrm{ES}_\alpha}(P, A_i).\end{aligned}$$

□

Correlation Bounds. *Shortfall correlation is bounded above and below as follows:*

$$-\frac{\mathrm{ES}(-A_i)}{\mathrm{ES}(A_i)} \le \rho^{\mathrm{ES}}(P, A_i) \le 1.$$

Expected shortfall is a convex risk measure which means that the marginal contribution of the expected shortfall of a portfolio $\mathrm{ES}_\alpha(P)$ with respect to every asset weight is monotonic. In other words, if the marginal contribution is evaluated at two points $x_i = a$ and $x_i = c$, the value of the gradient at an intermediate point $x_i = b$ is bounded:

$$\left.\frac{\partial \mathrm{ES}_\alpha(P)}{\partial x_i}\right|_{x_i=a} \le \left.\frac{\partial \mathrm{ES}_\alpha(P)}{\partial x_i}\right|_{x_i=b} \le \left.\frac{\partial \mathrm{ES}_\alpha(P)}{\partial x_i}\right|_{x_i=c} \tag{8}$$

Formula (8) tells us, that to bound the gradient, we need to consider the largest and smallest possible values of asset weight x_i, which are: $a = -\infty$ and $c = +\infty$. From Formula (8), for *any* asset weight $x_i = b$, we have the bounds:

$$\left.\frac{\partial \mathrm{ES}_\alpha(P)}{\partial x_i}\right|_{x_i=-\infty} \le \left.\frac{\partial \mathrm{ES}_\alpha(P)}{\partial x_i}\right|_{x_i=b} \le \left.\frac{\partial \mathrm{ES}_\alpha(P)}{\partial x_i}\right|_{x_i=+\infty} \tag{9}$$

As $x_i \to +\infty$, the portfolio $P_{x_i=+\infty}$ tends to a portfolio composed solely of asset A_i. The expected shortfall of this portfolio is:

$$\lim_{x_i \to +\infty} \mathrm{ES}_\alpha(P) = x_i \mathrm{ES}_\alpha(A_i) \tag{10}$$

where we have used the fact that expected shortfall is positively linear homogeneous in assets weights. Therefore:

$$\left.\frac{\partial \mathrm{ES}_\alpha(P)}{\partial x_i}\right|_{x_i=+\infty} = \mathrm{ES}_\alpha(A_i) \tag{11}$$

Similarly, the same gradient evaluated at $x_i \to -\infty$ is:

$$\left.\frac{\partial \mathrm{ES}_\alpha(P)}{\partial x_i}\right|_{x_i=-\infty} = -\mathrm{ES}_\alpha(-A_i) \tag{12}$$

where $\mathrm{ES}_\alpha(-A_i)$ is the expected shortfall of a short position in asset A_i. Using Formula (11) and Formula (12) in Formula (9) results in the bounds:

$$-\mathrm{ES}_\alpha(-A_i) \leq \left.\frac{\partial \mathrm{ES}_\alpha(P)}{\partial x_i}\right|_{x_i=b} \leq \mathrm{ES}_\alpha(A_i) \tag{13}$$

Dividing throughout by $\mathrm{ES}_\alpha(A_i)$ and using the shortfall correlation formula results in the bounds for shortfall correlation:

$$-\frac{\mathrm{ES}_\alpha(-A_i)}{\mathrm{ES}_\alpha(A_i)} \leq \rho^{\mathrm{ES}_\alpha}(P, A_i) \leq 1 \tag{14}$$

If the return distribution of asset A_i is symmetrical the bounds in Formula (14) simplify to:

$$-1 \leq \rho^{\mathrm{ES}_\alpha}(P, A_i) \leq 1 \tag{15}$$

Note that for Formula (15) we do not require that the symmetric distribution be normally distributed.

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