Risk decomposition using Euler’s theorem

Let $\text{RM}_p(w)$ denote a portfolio risk measure that is a homogenous function of degree one in the portfolio weight vector $w$. Euler’s theorem gives the additive risk decomposition

$$\text{RM}_p(w) = \sum_{i=1}^{n} w_i \frac{\partial \text{RM}_p(w)}{\partial w_i}$$

$$= w' \frac{\partial \text{RM}_p(w)}{\partial w}$$
Terminology:

\[ \text{MCR}^R_{i}^{M} = \frac{\partial \text{RM}_p(w)}{\partial w_i} \] = marginal contribution to risk of asset i,

\[ \text{CR}^R_{i}^{M} = w_i \cdot \text{MCR}^R_{i}^{M} \] = contribution to risk of asset i,

\[ \text{PCR}^R_{i}^{M} = \frac{\text{CR}^R_{i}^{M}}{\text{RM}_p(w)} \] = percent contribution of asset i

Then

\[ \text{RM}_p(w) = w_1 \cdot \text{MCR}^R_{1}^{M} + w_2 \cdot \text{MCR}^R_{2}^{M} + \cdots + w_n \cdot \text{MCR}^R_{n}^{M} \]

\[ = \text{CR}^R_{1}^{M} + \text{CR}^R_{2}^{M} + \cdots + \text{CR}^R_{n}^{M} \]

\[ 1 = \frac{\text{CR}^R_{1}^{M}}{\text{RM}_p(w)} + \cdots + \frac{\text{CR}^R_{n}^{M}}{\text{RM}_p(w)} = \text{PCR}^R_{1}^{M} + \cdots + \text{PCR}^R_{n}^{M}, \]
Risk Decomposition for Portfolio SD

\[ RM_p(w) = \sigma_p(w) = (w'\Sigma w)^{1/2} = w'\frac{\partial \sigma_p(w)}{\partial w} \]

Now

\[ \frac{\partial \sigma_p(w)}{\partial w} = \frac{\partial (w'\Sigma w)^{1/2}}{\partial w} = \frac{1}{2}(w'\Sigma w)^{-1/2}2\Sigma w \]
\[ = \frac{\Sigma w}{(w'\Sigma w)^{1/2}} = \frac{\Sigma w}{\sigma_p(w)} \]
\[ \Rightarrow \frac{\partial \sigma_p(w)}{\partial w_i} = \text{ith row of } \frac{(\Sigma x)}{\sigma_p(w)} \]

Remark: In R, the PerformanceAnalytics function \( \text{StdDev}(\cdot) \) performs this decomposition based on the sample covariance matrix of returns.
Example: 2 asset portfolio

\[ \sigma_p(w) = (w' \Sigma w)^{1/2} = (w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1w_2\sigma_{12})^{1/2} \]

\[ \Sigma w = \left( \begin{array}{cc} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{array} \right) \left( \begin{array}{c} w_1 \\ w_2 \end{array} \right) = \left( \begin{array}{c} w_1\sigma_1^2 + w_2\sigma_{12} \\ w_2\sigma_2^2 + w_1\sigma_{12} \end{array} \right) \]

\[ \frac{\Sigma w}{\sigma_p(w)} = \left( \begin{array}{c} \frac{w_1\sigma_1^2 + w_2\sigma_{12}}{\sigma_p(w)} \\ \frac{w_2\sigma_2^2 + w_1\sigma_{12}}{\sigma_p(w)} \end{array} \right) \]
Then

\[ \text{MCR}_1^\sigma = \left( w_1\sigma_1^2 + w_2\sigma_{12} \right) / \sigma_p(w) \]
\[ \text{MCR}_2^\sigma = \left( w_2\sigma_2^2 + w_1\sigma_{12} \right) / \sigma_p(w) \]
\[ \text{CR}_1^\sigma = w_1 \times \left( w_1\sigma_1^2 + w_2\sigma_{12} \right) / \sigma_p(w) = \left( w_1^2\sigma_1^2 + w_1w_2\sigma_{12} \right) / \sigma_p(w) \]
\[ \text{CR}_2^\sigma = w_2 \times \left( w_2\sigma_2^2 + w_2\sigma_{12} \right) / \sigma_p(w) = \left( w_2^2\sigma_2^2 + w_1w_2\sigma_{12} \right) / \sigma_p(w) \]

and

\[ \text{PCR}_1^\sigma = \text{CR}_1^\sigma / \sigma_p(w) = \left( w_1^2\sigma_1^2 + w_1w_2\sigma_{12} \right) / \sigma_p^2(w) \]
\[ \text{PCR}_2^\sigma = \text{CR}_2^\sigma / \sigma_p(w) = \left( w_2^2\sigma_2^2 + w_1w_2\sigma_{12} \right) / \sigma_p^2(w) \]
How to Interpret and Use $MCR_i^\sigma$

$$MCR_i^\sigma = \frac{\partial \sigma_p(w)}{\partial w_i} \approx \frac{\Delta \sigma_p}{\Delta w_i}$$

$$\Rightarrow \Delta \sigma_p \approx MCR_i^\sigma \cdot \Delta w_i$$

However, in a portfolio of $n$ assets

$$w_1 + w_2 + \cdots + w_n = 1$$

so that increasing or decreasing $w_i$ means that we have to decrease or increase our allocation to one or more other assets. Hence, the formula

$$\Delta \sigma_p \approx MCR_i^\sigma \cdot \Delta w_i$$

ignores this re-allocation effect.
If the increase in allocation to asset $i$ is offset by a decrease in allocation to asset $j$, then

$$\Delta w_j = -\Delta w_i$$

and the change in portfolio volatility is approximately

$$\Delta \sigma_p \approx \text{MCR}_i^{\sigma} \cdot \Delta w_i + \text{MCR}_j^{\sigma} \cdot \Delta w_j$$

$$= \text{MCR}_i^{\sigma} \cdot \Delta w_i - \text{MCR}_j^{\sigma} \cdot \Delta w_i$$

$$= \left(\text{MCR}_i^{\sigma} - \text{MCR}_j^{\sigma}\right) \cdot \Delta w_i$$
Consider two portfolios:

- equal weighted portfolio $w_1 = w_2 = 0.5$
- long-short portfolio $w_1 = 1.5$ and $w_2 = -0.5$. 

<table>
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<tr>
<th>$\mu_1$</th>
<th>$\mu_2$</th>
<th>$\sigma^2_1$</th>
<th>$\sigma^2_2$</th>
<th>$\sigma_1$</th>
<th>$\sigma_2$</th>
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<td>-0.164</td>
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Table 1: Example data for two asset portfolio.
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<tr>
<td>Asset 1</td>
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<td>0.5</td>
<td>0.23310</td>
<td>0.11655</td>
<td>0.8807</td>
</tr>
<tr>
<td>Asset 2</td>
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<td>Asset 2</td>
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<td>-0.03474</td>
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<td>0.04337</td>
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</tbody>
</table>

Table 2: Risk decomposition using portfolio standard deviation.

**Interpretation:** For equally weighted portfolio, increasing $w_1$ from 0.5 to 0.6 decreases $w_2$ from 0.5 to 0.4. Then

$$\Delta \sigma_p \approx (\text{MCR}^{\sigma}_1 - \text{MCR}^{\sigma}_2) \cdot \Delta w_i$$

$$= (0.23310 - 0.03158)(0.1)$$

$$= 0.02015$$
For the long-short portfolio, increasing $w_1$ from 1.5 to 1.6 decreases $w_2$ from -0.5 to -0.6. Then

$$\Delta \sigma_p \approx (\text{MCR}_1^\sigma - \text{MCR}_2^\sigma) \cdot \Delta w_i$$

$$= [0.25540 - (-0.03474)] (0.1)$$

$$= 0.02901$$
Remark

The reallocation does not have to be all in one asset. Suppose, the reallocation is spread across all of the other assets $j \neq i$ so that

$$\Delta w_j = -\alpha_j \Delta w_i \text{ s.t. } \sum_{j \neq i} \alpha_j = 1$$

Then

$$\sum_{j \neq i} \Delta w_j = -\sum_{j \neq i} \alpha_j \Delta w_i = -\Delta w_i \sum_{j \neq i} \alpha_j = -\Delta w_i$$

and

$$\Delta \sigma_p \approx \text{MCR}_i^\sigma \cdot \Delta w_i + \sum_{j \neq i} \text{MCR}_j^\sigma \cdot \Delta w_j$$

$$= \left( \text{MCR}_i^\sigma - \sum_{j \neq i} \alpha_j \cdot \text{MCR}_j^\sigma \right) \Delta w_i$$
In matrix notation the result can be written as

\[ \Delta \sigma_p \approx (\text{MCR}^\sigma' \alpha) \Delta w_i \]

\[ \text{MCR}^\sigma = (\text{MCR}_1^\sigma, \ldots, \text{MCR}_n^\sigma)' \]

\[ \alpha = (-\alpha_1, \ldots, -\alpha_{i-1}, 1, -\alpha_{i+1}, \ldots -\alpha_n)' \]
Interpreting MCR when $RM(w) = VaR_\alpha(w)$ or $ES_\alpha(w)$

- $\Delta w_j = -\Delta w_i$

\[
\Delta VaR_\alpha(w) = \left( \text{MCR}_i^{VaR_\alpha} - \text{MCR}_j^{VaR_\alpha} \right) \cdot \Delta w_i
\]

\[
\Delta ES_\alpha(w) = \left( \text{MCR}_i^{ES_\alpha} - \text{MCR}_j^{ES_\alpha} \right) \cdot \Delta w_i
\]

- $\Delta w_j = -\alpha_j \Delta w_i$ s.t. $\sum_{j \neq i} \alpha_j = 1$

\[
\Delta VaR_\alpha(w) \approx \left( \text{MCR}_i^{VaR_\alpha' \alpha} \right) \Delta w_i
\]

\[
\Delta ES_\alpha(w) \approx \left( \text{MCR}_i^{VaR_\alpha' \alpha} \right) \Delta w_i
\]
Beta as a Measure of Asset Contribution to Portfolio Volatility

For a portfolio of $n$ assets with return

$$R_p(w) = w_1 R_1 + \cdots + w_n R_n = w'R$$

we derived the portfolio volatility decomposition

$$\sigma_p(w) = w_1 \frac{\partial \sigma_p(w)}{\partial w_1} + w_2 \frac{\partial \sigma_p(w)}{\partial w_2} + \cdots + w_n \frac{\partial \sigma_p(w)}{\partial w_n} = w' \frac{\partial \sigma_p(w)}{\partial w}$$

$$\frac{\partial \sigma_p(w)}{\partial w} = \frac{\Sigma w}{\sigma_p(w)} \frac{\partial \sigma_p(w)}{\partial w_i} = \text{ith row of} \frac{\Sigma w}{\sigma_p(w)}$$

With a little bit of algebra we can derive an alternative expression for

$$\text{MCR}^\sigma_i = \frac{\partial \sigma_p(w)}{\partial w_i} = \text{ith row of} \frac{\Sigma w}{\sigma_p(w)}$$
**Definition:** The beta of asset $i$ with respect to the portfolio is defined as

$$\beta_i = \frac{\text{cov}(R_i, R_p(w))}{\text{var}(R_p(w))} = \frac{\text{cov}(R_i, R_p(w))}{\sigma_p^2(w)}$$

**Result:** $\beta_i$ measures asset contribution to $\sigma_p(w)$:

$$\text{MCR}_i^\sigma = \frac{\partial \sigma_p(w)}{\partial w_i} = \beta_i \sigma_p(w)$$
$$\text{CR}_i^\sigma = w_i \beta_i \sigma_p(w)$$
$$\text{PCR}_i^\sigma = w_i \beta_i$$
Remarks

• By construction, the beta of the portfolio is 1

$$\beta_p = \frac{\text{cov}(R_p(w), R_p(w))}{\text{var}(R_p(w))} = \frac{\text{var}(R_p(w))}{\text{var}(R_p(w))} = 1$$

• When $\beta_i = 1$

$$\text{MCR}_i^\sigma = \sigma_p(w)$$
$$\text{CR}_i^\sigma = w_i \sigma_p(w)$$
$$\text{PCR}_i^\sigma = w_i$$
• When $\beta_i > 1$

$$MCR^\sigma_i > \sigma_p(\mathbf{w})$$
$$CR^\sigma_i > w_i \sigma_p(\mathbf{w})$$
$$PCR^\sigma_i > w_i$$

• When $\beta_i < 1$

$$MCR^\sigma_i < \sigma_p(\mathbf{w})$$
$$CR^\sigma_i < w_i \sigma_p(\mathbf{w})$$
$$PCR^\sigma_i < w_i$$
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Table 3: Risk decomposition using portfolio standard deviation.

Remarks:

- For the equally weighted portfolio, Asset 1 is a risk enhancer and Asset 2 is a risk reducer.
- For the long-short portfolio, $\beta_2 < 0$!
$w - \sigma - \rho$ Decomposition of Portfolio Volatility

Recall,

$$MCR_i^\sigma = \frac{\partial \sigma_p(w)}{\partial w_i} = \text{ith row of } \frac{\Sigma w}{\sigma_p(w)} = \frac{\text{cov}(R_i, R_p(w))}{\sigma_p(w)}$$

Using

$$\rho_{i,p} = \text{corr}(R_i, R_p(w)) = \frac{\text{cov}(R_i, R_p(w))}{\sigma_i \sigma_p(w)}$$

$$\Rightarrow \text{cov}(R_i, R_p(w)) = \rho_{i,p} \sigma_i \sigma_p(w)$$

gives

$$MCR_i^\sigma = \frac{\rho_{i,p} \sigma_i \sigma_p(w)}{\sigma_p(w)} = \rho_{i,p} \sigma_i$$
Then

\[ CR_i^\sigma = w_i \times MCR_i^\sigma = w_i \times \sigma_i \times \rho_{i,p} \]

= allocation \times standalone risk \times correlation with portfolio

Remarks:

- \( w_i \times \sigma_i \) = standalone contribution to risk (ignores correlation effects with other assets)

- \( CR_i^\sigma = w_i \times \sigma_i \) only when \( \rho_{i,p} = 1 \)

- If \( \rho_{i,p} \neq 1 \) then \( CR_i^\sigma < w_i \times \sigma_i \)
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<tr>
<th></th>
<th>$\sigma_i$</th>
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</tbody>
</table>

Table 4: Risk decomposition using portfolio standard deviation.

Remarks:

- For the equally weighted portfolio, both assets are positively correlated with the portfolio
- For the long-short portfolio, Asset 2 is negatively correlated with the portfolio
Remark:

- Goldberg, Hayes, Menchero and Mitra (2009) “Extreme Risk Management”, Barra Technical Report, extends the $\beta$ and $w - \sigma - \rho$ decompositions to ES
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<tr>
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Table 5: Portfolio VaR Report

**Portfolio VaR and ES Reports**

A common portfolio risk report summarizes asset and portfolio risk measures as well as risk budgets.