Amath 546/Econ 589
Risk Budgeting and Risk Reporting

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Risk decomposition using Euler's theorem

Let $\mathsf{RM}_p(\mathbf{w})$ denote a portfolio risk measure that is a homogenous function of degree one in the portfolio weight vector $\mathbf{w}$. Euler's theorem gives the additive risk decomposition

$$\begin{aligned}
\mathsf{RM}_p(\mathbf{w}) &= w_1 \frac{\partial \mathsf{RM}_p(\mathbf{w})}{\partial w_1} + w_2 \frac{\partial \mathsf{RM}_p(\mathbf{w})}{\partial w_2} + \cdots + w_n \frac{\partial \mathsf{RM}_p(\mathbf{w})}{\partial w_n} \\
&= \sum_{i=1}^{n} w_i \frac{\partial \mathsf{RM}_p(\mathbf{w})}{\partial w_i} \\
&= \mathbf{w}' \frac{\partial \mathsf{RM}_p(\mathbf{w})}{\partial \mathbf{w}}
\end{aligned}$$

Terminology:

$$\begin{aligned}
\mathsf{MCR}_i^{RM} &= \frac{\partial \mathsf{RM}_p(\mathbf{w})}{\partial w_i} = \text{ marginal contribution to risk of asset i,} \\
\mathsf{CR}_i^{RM} &= w_i \cdot \mathsf{MCR}_i^{RM} = \text{ contribution to risk of asset i,} \\
\mathsf{PCR}_i^{RM} &= \frac{\mathsf{CR}_i^{RM}}{\mathsf{RM}_p(\mathbf{w})} = \text{ percent contribution of asset i}
\end{aligned}$$

Then

$$\begin{aligned}
\mathsf{RM}_p(\mathbf{w}) &= w_1 \cdot \mathsf{MCR}_1^{RM} + w_2 \cdot \mathsf{MCR}_2^{RM} + \cdots + w_n \cdot \mathsf{MCR}_n^{RM} \\
&= \mathsf{CR}_1^{RM} + \mathsf{CR}_2^{RM} + \cdots + \mathsf{CR}_n^{RM} \\
1 &= \frac{\mathsf{CR}_1^{RM}}{\mathsf{RM}_p(\mathbf{w})} + \cdots + \frac{\mathsf{CR}_n^{RM}}{\mathsf{RM}_p(\mathbf{w})} = \mathsf{PCR}_1^{RM} + \cdots + \mathsf{PCR}_n^{RM},
\end{aligned}$$

Risk Decomposition for Portfolio SD

$$\mathsf{RM}_p(\mathbf{w}) = \sigma_p(\mathbf{w}) = (\mathbf{w}'\mathbf{\Sigma}\mathbf{w})^{1/2} = \mathbf{w}'\frac{\partial \sigma_p(\mathbf{w})}{\partial \mathbf{w}}$$

Now

$$\begin{aligned}
\frac{\partial \sigma_p(\mathbf{w})}{\partial \mathbf{w}} &= \frac{\partial (\mathbf{w}'\mathbf{\Sigma}\mathbf{w})^{1/2}}{\partial \mathbf{w}} = \frac{1}{2}(\mathbf{w}'\mathbf{\Sigma}\mathbf{w})^{-1/2} 2\mathbf{\Sigma}\mathbf{w} \\
&= \frac{\mathbf{\Sigma}\mathbf{w}}{(\mathbf{w}'\mathbf{\Sigma}\mathbf{w})^{1/2}} = \frac{\mathbf{\Sigma}\mathbf{w}}{\sigma_p(\mathbf{w})} \\
\Rightarrow &\quad \frac{\partial \sigma_p(\mathbf{w})}{\partial w_i} = \text{ith row of } \frac{\mathbf{\Sigma}\mathbf{x}}{\sigma_p(\mathbf{w})}
\end{aligned}$$

Remark: In R, the PerformanceAnalytics function `StdDev()` performs this decomposition based on the sample covariance matrix of returns.

Example: 2 asset portfolio

$$\begin{aligned}
\sigma_p(\mathbf{w}) &= (\mathbf{w}'\mathbf{\Sigma}\mathbf{w})^{1/2} = \left(w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}\right)^{1/2} \\
\mathbf{\Sigma}\mathbf{w} &= \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1\sigma_1^2 + w_2\sigma_{12} \\ w_2\sigma_2^2 + w_1\sigma_{12} \end{pmatrix} \\
\frac{\mathbf{\Sigma}\mathbf{w}}{\sigma_p(\mathbf{w})} &= \begin{pmatrix} \left(w_1\sigma_1^2 + w_2\sigma_{12}\right)/\sigma_p(\mathbf{w}) \\ \left(w_2\sigma_2^2 + w_1\sigma_{12}\right)/\sigma_p(\mathbf{w}) \end{pmatrix}
\end{aligned}$$

Then

$$\begin{aligned}
\mathsf{MCR}_1^\sigma &= \left(w_1\sigma_1^2 + w_2\sigma_{12}\right)/\sigma_p(\mathbf{w}) \\
\mathsf{MCR}_2^\sigma &= \left(w_2\sigma_2^2 + w_1\sigma_{12}\right)/\sigma_p(\mathbf{w}) \\
\mathsf{CR}_1^\sigma &= w_1 \times \left(w_1\sigma_1^2 + w_2\sigma_{12}\right)/\sigma_p(\mathbf{w}) = \left(w_1^2\sigma_1^2 + w_1w_2\sigma_{12}\right)/\sigma_p(\mathbf{w}) \\
\mathsf{CR}_2^\sigma &= w_2 \times \left(w_2\sigma_2^2 + w_2\sigma_2\right)/\sigma_p(\mathbf{w}) = \left(w_2^2\sigma_2^2 + w_1w_2\sigma_{12}\right)/\sigma_p(\mathbf{w})
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{PCR}_1^\sigma &= \mathsf{CR}_1^\sigma/\sigma_p(\mathbf{w}) = \left(w_1^2\sigma_1^2 + w_1w_2\sigma_{12}\right)/\sigma_p^2(\mathbf{w}) \\
\mathsf{PCR}_2^\sigma &= \mathsf{CR}_2^\sigma/\sigma_p(\mathbf{w}) = \left(w_2^2\sigma_2^2 + w_1w_2\sigma_{12}\right)/\sigma_p^2(\mathbf{w})
\end{aligned}$$

How to Interpret and Use MCR_i^{σ}

$$\begin{aligned}\mathsf{MCR}_i^{\sigma} &= \frac{\partial \sigma_p(\mathbf{w})}{\partial w_i} \approx \frac{\Delta \sigma_p}{\Delta w_i} \\ &\Rightarrow \Delta \sigma_p \approx \mathsf{MCR}_i^{\sigma} \cdot \Delta w_i\end{aligned}$$

However, in a portfolio of n assets

$$w_1 + w_2 + \cdots + w_n = 1$$

so that increasing or decreasing w_i means that we have to decrease or increase our allocation to one or more other assets. Hence, the formula

$$\Delta \sigma_p \approx \mathsf{MCR}_i^{\sigma} \cdot \Delta w_i$$

ignores this re-allocation effect.

If the increase in allocation to asset i is offset by a decrease in allocation to asset j, then

$$\Delta w_j = -\Delta w_i$$

and the change in portfolio volatility is approximately

$$\begin{aligned}
\Delta\sigma_p &\approx \mathsf{MCR}_i^\sigma \cdot \Delta w_i + \mathsf{MCR}_j^\sigma \cdot \Delta w_j \\
&= \mathsf{MCR}_i^\sigma \cdot \Delta w_i - \mathsf{MCR}_j^\sigma \cdot \Delta w_i \\
&= \left(\mathsf{MCR}_i^\sigma - \mathsf{MCR}_j^\sigma\right) \cdot \Delta w_i
\end{aligned}$$

μ_1	μ_2	σ_1^2	σ_2^2	σ_1	σ_2	σ_{12}	ρ_{12}
0.175	0.055	0.067	0.013	0.258	0.115	-0.004875	-0.164

Table 1: Example data for two asset portfolio.

Consider two portfolios:

- equal weighted portfolio $w_1 = w_2 = 0.5$
- long-short portfolio $w_1 = 1.5$ and $w_2 = -0.5$.

	σ_i	w_i	MCR_i^σ	CR_i^σ	PCR_i^σ
			$\sigma_p = 0.1323$		
Asset 1	0.258	0.5	0.23310	0.11655	0.8807
Asset 2	0.115	0.5	0.03158	0.01579	0.1193
			$\sigma_p = 0.4005$		
Asset 1	0.258	1.5	0.25540	0.38310	0.95663
Asset 2	0.115	-0.5	-0.03474	0.01737	0.04337

Table 2: Risk decomposition using portfolio standard deviation.

Interpretation: For equally weighted portfolio, increasing w_1 from 0.5 to 0.6 decreases w_2 from 0.5 to 0.4. Then

$$\begin{aligned}\Delta\sigma_p &\approx (\text{MCR}_1^\sigma - \text{MCR}_2^\sigma)\cdot\Delta w_i\\ &= (0.23310 - 0.03158)(0.1)\\ &= 0.02015\end{aligned}$$

For the long-short portfolio, increasing w_1 from 1.5 to 1.6 decreases w_2 from -0.5 to -0.6. Then

$$\begin{aligned}\Delta\sigma_p &\approx (\mathsf{MCR}_1^{\sigma} - \mathsf{MCR}_2^{\sigma}) \cdot \Delta w_i \\ &= [0.25540 - (-0.03474)]\,(0.1) \\ &= 0.02901\end{aligned}$$

Remark

The reallocation does not have to be all in one asset. Suppose, the reallocation is spread across all of the other assets $j \neq i$ so that

$$\Delta w_j = -\alpha_j \Delta w_i \text{ s.t. } \sum_{j \neq i} \alpha_j = 1$$

Then

$$\sum_{j \neq i} \Delta w_j = -\sum_{j \neq i} \alpha_j \Delta w_i = -\Delta w_i \sum_{j \neq i} \alpha_j = -\Delta w_i$$

and

$$\begin{aligned} \Delta \sigma_p &\approx \mathsf{MCR}_i^{\sigma} \cdot \Delta w_i + \sum_{j \neq i} \mathsf{MCR}_j^{\sigma} \cdot \Delta w_j \\ &= \left(\mathsf{MCR}_i^{\sigma} \cdot - \sum_{j \neq i} \alpha_j \cdot \mathsf{MCR}_j^{\sigma} \right) \Delta w_i \end{aligned}$$

In matrix notation the result can be written as

$$
\begin{aligned}
\boldsymbol{\Delta}\sigma_p &\approx \left(\mathbf{MCR}^{\sigma\prime}\boldsymbol{\alpha}\right)\boldsymbol{\Delta}w_i \\
\mathbf{MCR}^{\sigma} &= (\mathsf{MCR}_1^{\sigma}, \ldots, \mathsf{MCR}_n^{\sigma})' \\
\boldsymbol{\alpha} &= (-\alpha_1, \ldots, -\alpha_{i-1}, 1, -\alpha_{i+1}, \ldots - \alpha_n)'
\end{aligned}
$$

Interpreting MCR when $RM(\mathbf{w}) = VaR_\alpha(\mathbf{w})$ or $ES_\alpha(\mathbf{w})$

- $\Delta w_j = -\Delta w_i$

$$\begin{aligned} \Delta VaR_\alpha(\mathbf{w}) &= \left(\mathsf{MCR}_i^{VaR_\alpha} - \mathsf{MCR}_j^{VaR_\alpha}\right) \cdot \Delta w_i \\ \Delta ES_\alpha(\mathbf{w}) &= \left(\mathsf{MCR}_i^{ES_\alpha} - \mathsf{MCR}_j^{ES_\alpha}\right) \cdot \Delta w_i \end{aligned}$$

- $\Delta w_j = -\alpha_j \Delta w_i$ s.t. $\sum_{j \neq i} \alpha_j = 1$

$$\begin{aligned} \Delta VaR_\alpha(\mathbf{w}) &\approx \left(\mathbf{MCR}^{VaR_\alpha \prime} \boldsymbol{\alpha}\right) \Delta w_i \\ \Delta ES_\alpha(\mathbf{w}) &\approx \left(\mathbf{MCR}^{VaR_\alpha \prime} \boldsymbol{\alpha}\right) \Delta w_i \end{aligned}$$

Beta as a Measure of Asset Contribution to Portfolio Volatility

For a portfolio of n assets with return

$$R_p(\mathbf{w}) = w_1 R_1 + \cdots + w_n R_n = \mathbf{w}'\mathbf{R}$$

we derived the portfolio volatility decomposition

$$\begin{aligned} \sigma_p(\mathbf{w}) &= w_1 \frac{\partial \sigma_p(\mathbf{w})}{\partial w_1} + w_2 \frac{\partial \sigma_p(\mathbf{w})}{\partial w_2} + \cdots + w_n \frac{\partial \sigma_p(\mathbf{w})}{\partial w_n} = \mathbf{w}' \frac{\partial \sigma_p(\mathbf{w})}{\partial \mathbf{w}} \\ \frac{\partial \sigma_p(\mathbf{w})}{\partial \mathbf{w}} &= \frac{\mathbf{\Sigma w}}{\sigma_p(\mathbf{w})}, \ \frac{\partial \sigma_p(\mathbf{w})}{\partial w_i} = \text{ith row of } \frac{\mathbf{\Sigma w}}{\sigma_p(\mathbf{w})} \end{aligned}$$

With a little bit of algebra we can derive an alternative expression for

$$\text{MCR}_i^{\sigma} = \frac{\partial \sigma_p(\mathbf{w})}{\partial w_i} = \text{ith row of } \frac{\mathbf{\Sigma w}}{\sigma_p(\mathbf{w})}$$

Definition: The beta of asset i with respect to the portfolio is defined as

$$\beta_i = \frac{cov(R_i, R_p(\mathbf{w}))}{var(R_p(\mathbf{w}))} = \frac{cov(R_i, R_p(\mathbf{w}))}{\sigma_p^2(\mathbf{w})}$$

Result: β_i measures asset contribution to $\sigma_p(\mathbf{w})$:

$$\begin{aligned}
\mathsf{MCR}_i^{\sigma} &= \frac{\partial \sigma_p(\mathbf{w})}{\partial w_i} = \beta_i \sigma_p(\mathbf{w}) \\
\mathsf{CR}_i^{\sigma} &= w_i \beta_i \sigma_p(\mathbf{w}) \\
\mathsf{PCR}_i^{\sigma} &= w_i \beta_i
\end{aligned}$$

Remarks

- By construction, the beta of the portfolio is 1

$$\beta_p = \frac{cov(R_p(\mathbf{w}), R_p(\mathbf{w}))}{var(R_p(\mathbf{w}))} = \frac{var(R_p(\mathbf{w}))}{var(R_p(\mathbf{w}))} = 1$$

- When $\beta_i = 1$

$$\begin{aligned} \mathsf{MCR}_i^\sigma &= \sigma_p(\mathbf{w}) \\ \mathsf{CR}_i^\sigma &= w_i\sigma_p(\mathbf{w}) \\ \mathsf{PCR}_i^\sigma &= w_i \end{aligned}$$

- When $\beta_i > 1$

$$\begin{aligned} \mathsf{MCR}_i^{\sigma} &> \sigma_p(\mathbf{w}) \\ \mathsf{CR}_i^{\sigma} &> w_i \sigma_p(\mathbf{w}) \\ \mathsf{PCR}_i^{\sigma} &> w_i \end{aligned}$$

- When $\beta_i < 1$

$$\begin{aligned} \mathsf{MCR}_i^{\sigma} &< \sigma_p(\mathbf{w}) \\ \mathsf{CR}_i^{\sigma} &< w_i \sigma_p(\mathbf{w}) \\ \mathsf{PCR}_i^{\sigma} &< w_i \end{aligned}$$

	σ_i	w_i	β_i	MCR_i^σ	CR_i^σ	PCR_i^σ
			$\sigma_p = 0.1323$			
Asset 1	0.258	0.5	1.76	0.23310	0.11655	0.8807
Asset 2	0.115	0.5	0.24	0.03158	0.01579	0.1193
			$\sigma_p = 0.4005$			
Asset 1	0.258	1.5	0.64	0.25540	0.38310	0.95663
Asset 2	0.115	-0.5	-0.09	-0.03474	0.01737	0.04337

Table 3: Risk decomposition using portfolio standard deviation.

Remarks:

- For the equally weighted portfolio, Asset 1 is a risk enhancer and Asset 2 is a risk reducer
- For the long-short portfolio, $\beta_2 < 0$!

$w - \sigma - \rho$ Decomposition of Portfolio Volatility

Recall,

$$\text{MCR}_i^{\sigma} = \frac{\partial \sigma_p(\mathbf{w})}{\partial w_i} = \text{ith row of } \frac{\mathbf{\Sigma w}}{\sigma_p(\mathbf{w})} = \frac{cov(R_i, R_p(\mathbf{w}))}{\sigma_p(\mathbf{w})}$$

Using

$$\begin{aligned} \rho_{i,p} &= corr(R_i, R_p(\mathbf{w})) = \frac{cov(R_i, R_p(\mathbf{w}))}{\sigma_i \sigma_p(\mathbf{w})} \\ &\Rightarrow cov(R_i, R_p(\mathbf{w})) = \rho_{i,p}\sigma_i\sigma_p(\mathbf{w}) \end{aligned}$$

gives

$$\text{MCR}_i^{\sigma} = \frac{\rho_{i,p}\sigma_i\sigma_p(\mathbf{w})}{\sigma_p(\mathbf{w})} = \rho_{i,p}\sigma_i$$

Then

$$\mathsf{CR}_i^\sigma = w_i \times \mathsf{MCR}_i^\sigma = w_i \times \sigma_i \times \rho_{i,p}$$
$$= \text{allocation} \times \text{standalone risk} \times \text{correlation with portfolio}$$

Remarks:

- $w_i \times \sigma_i =$ standalone contribution to risk (ignores correlation effects with other assets)
- $\mathsf{CR}_i^\sigma = w_i \times \sigma_i$ only when $\rho_{i,p} = 1$
- If $\rho_{i,p} \neq 1$ then $\mathsf{CR}_i^\sigma < w_i \times \sigma_i$

	σ_i	w_i	ρ_i	MCR_i^σ	CR_i^σ	PCR_i^σ
			$\sigma_p = 0.1323$			
Asset 1	0.258	0.5	0.90	0.23310	0.11655	0.8807
Asset 2	0.115	0.5	0.27	0.03158	0.01579	0.1193
			$\sigma_p = 0.4005$			
Asset 1	0.258	1.5	0.99	0.25540	0.38310	0.95663
Asset 2	0.115	-0.5	-0.30	-0.03474	0.01737	0.04337

Table 4: Risk decomposition using portfolio standard deviation.

Remarks:

- For the equally weighted portfolio, both assets are positively correlated with the portfolio
- For the long-short portfolio, Asset 2 is negatively correlated with the portfolio

Remark:

- Goldberg, Hayes, Menchero and Mitra (2009) "Extreme Risk Management", Barra Technical Report, extends the β and $w - \sigma - \rho$ decompositions to ES

Asset	**$**	w_i	μ	σ	Asset VaR	MCVaR	CVaR	PCVaR
Asset 1	10	.10	.01	.10	-.03	.003	.01	.10
Asset 2	20	.20	.02	.12	-.04	.002	.02	.11
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Asset N	5	.05	.01	.07	-.07	.010	.04	.13
Portfolio	100	1	.03	.08			.08	1

Table 5: Portfolio VaR Report

Portfolio VaR and ES Reports

A common portfolio risk report summarizes asset and portfolio risk measures as well as risk budgets