AMATH 546/ECON 589
Risk Measures

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Outline

- Profit, Loss and Return Distributions
- Risk measures
- Risk measure properties
- Portfolio risk measures and risk budgeting

Reading

- FRF chapter 4
- QRM chapter 2, sections 1 and 2; chapter 6
- FMUND chapter 8
- SADFE chapter 19
**Profits and Losses**

- $V_t = \text{value (price) of an asset at time } t \text{ (assumed known)} \text{ measured in } $

- $V_{t+1} = \text{value (price) of an asset at time } t+1 \text{ (typically unknown)} \text{ measured in }$

- $\Pi_{t+1} = V_{t+1} - V_t = \text{profit measured in } $ \text{ over the holding period}$

- $L_{t+1} = -\Pi_{t+1} = \text{loss measured in } $ \text{ over the holding period}$

**Remarks**

- $\Pi_{t+1} > 0 \text{ (positive profit)} \implies L_{t+1} < 0 \text{ (negative loss)}$

- $\Pi_{t+1} < 0 \text{ (negative profit)} \implies L_{t+1} > 0 \text{ (positive loss)}$

- Risk measures are typically defined in terms of losses. Hence a large positive value for a risk measure indicates a large positive loss.
Returns

- \( R_{t+1} = \frac{V_{t+1} - V_t}{V_t} \) = simple return between times \( t \) and \( t + 1 \)

- \( \Pi_{t+1} = V_t R_{t+1} = V_{t+1} - V_t \)

- \( L_{t+1} = -V_t R_{t+1} \)

**Example** (do on white board)

Profit, Loss and Return Distributions

- \( \Pi_{t+1}, L_{t+1} \) and \( R_{t+1} \) are random variables because at time \( t \) the future value \( V_{t+1} \) is not known.

- For the random variable \( X = \Pi_{t+1}, L_{t+1} \) and \( R_{t+1} \), let \( F_X \) denote the CDF and \( f_X \) denote the pdf

- The distributions of \( \Pi_{t+1}, L_{t+1} \) and \( R_{t+1} \) are obviously linked

- Assume (unless otherwise specified) that \( F_X \) and \( f_X \) are are known and are continuous functions

**Example** (do on white board)
Risk Measurement

Goal: Make risk comparisons across a variety of assets to aid decision making of some sort.

- Determination of risk capital and capital adequacy
- Management tool
- Insurance premiums

Definition 1 (Risk measure) Mathematical method for capturing risk

Definition 2 (Risk measurement) A number that captures risk. It is obtained by applying data to a risk measure

Three Most Common Risk Measures

1. Volatility (vol or $\sigma$)

2. Value-at-Risk (VaR)

3. Expected Shortfall (ES)
   - aka Expected Tail Loss (ETL), conditional VaR (cVaR)
Volatility

Profit/Loss volatility

\[ \sigma_{\Pi} = \left( E[(\Pi_{t+1} - \mu_{\Pi})^2] \right)^{1/2} = \sigma_L = \left( E[(L_{t+1} - \mu_L)^2] \right)^{1/2} \]

Return Volatility

\[ \sigma_R = \left( E[(R_{t+1} - \mu_R)^2] \right)^{1/2} \]

Relationship between \( \sigma_L \) and \( \sigma_R \)

\[ L_{t+1} = V_t R_{t+1} \Rightarrow \sigma_L = V_t \sigma_R \]

Remarks

- volatility measures the size of a typical deviation from the mean loss or return.

- volatility is a symmetric risk measure - does not focus on downside risk.

- volatility is an appropriate risk measure if losses or returns have a normal distribution.

- volatility might not exist (i.e., might not be a finite number)
Value-at-Risk

Let $F_L$ denote the distribution of losses $L_{t+1}$ on some asset over a given holding period.

**Definition 3 (Value-at-Risk, QRM)** Given some confidence level $\alpha \in (0, 1)$. The VaR on our asset at the confidence level $\alpha$ is given by the smallest number $\lambda$ such that the probability that the loss $L_{t+1}$ exceeds $\lambda$ is no larger than $(1 - \alpha)$. Formally,

$$VaR_\alpha = \inf \{ \lambda \in \mathbb{R} : \Pr(L_{t+1} > \lambda) \leq 1 - \alpha \}$$

If $F_L$ is continuous then $VaR_\alpha$ can be implicitly defined using

$$\Pr(L_{t+1} \geq VaR_\alpha) = 1 - \alpha$$

or

$$F_L(VaR_\alpha) = \Pr(L_{t+1} \leq VaR_\alpha) = \alpha$$

Remarks

- $VaR_\alpha$ is the upper $\alpha$-quantile of the loss distribution $F_L$. If $F_L$ is continuous then $VaR_\alpha$ can be conveniently computed using the quantile function $F_L^{-1}$

$$VaR_\alpha = F_L^{-1}(\alpha) = q^L_\alpha$$

- Typically $\alpha = 0.90$, 0.95 or 0.99. If $\alpha = 0.95$ then with 95% confidence we could lose $VaR_{0.95}$ or more over the holding period.

- $VaR_\alpha$ is a *lower bound* on the possible losses that might occur with confidence level $\alpha$. It says nothing about the magnitude of loses beyond $VaR_\alpha$. 
Alternative Definitions of VaR

Unfortunately, there is no universally accepted definition of $VaR_\alpha$

- Some authors define $\alpha$ as the *probability of loss*. In this case, for continuous $F_L$, we have

  \[ \Pr(L_{t+1} \geq VaR_\alpha) = \alpha \] so that

  \[ VaR_\alpha = F_L^{-1}(1 - \alpha) = q_{1-\alpha}^L \]

  For example, if $\alpha = 0.05$, then with 5% probability we could lose $VaR_{0.05}$ or more over the holding period.

- Some authors (e.g. FRF) define $VaR$ using the distribution of profits. Since $\Pi_{t+1} = -L_{t+1}$ we have (using $\alpha$ as confidence level)

  \[ \Pr(L_{t+1} \geq VaR_\alpha) = 1 - \alpha \]

  \[ = \Pr(-\Pi_{t+1} \geq VaR_\alpha) = 1 - \alpha \]

  \[ = \Pr(\Pi_{t+1} \leq -VaR_\alpha) = 1 - \alpha \]

  Following FRF, if $VaR$ is defined using probability of loss then use $\alpha = p$ to denote loss probability and write $VaR_p$. Then

  \[ \Pr(\Pi_{t+1} \leq -VaR_p) = p \]
Some authors define $VaR$ using the distribution of returns. Since $L_{t+1} = -V_t R_{t+1}$ we have (using $\alpha$ as confidence level)

$$\Pr(L_{t+1} \geq VaR) = 1 - \alpha$$

$$= \Pr(-V_t R_{t+1} \geq VaR) = 1 - \alpha$$

$$= \Pr(-R_{t+1} \geq \frac{VaR}{V_t}) = 1 - \alpha$$

Here, $\frac{VaR}{V_t} = q^{-R}_\alpha$ is the upper $\alpha$-quantile of the (negative) returns.

**Steps to Calculating VaR**
Example: Calculating VaR when losses/returns follow a normal distribution

Let $L_{t+1} \sim N(\mu_L, \sigma_L^2)$ where $\mu_L$ and $\sigma_L$ are known. The pdf and CDF are given by

$$f_L(l; \mu_L, \sigma_L) = \frac{1}{\sqrt{2\pi\sigma_L^2}} e^{-\frac{1}{2}\left(\frac{l-\mu_L}{\sigma_L}\right)^2}$$

$$F_L(l; \mu_L, \sigma_L) = \Pr(L_{t+1} \leq l) = \int_{-\infty}^{l} f_L(x; \mu_L, \sigma_L) \, dx$$

Then, for a given confidence level $\alpha \in (0, 1)$

$$VaR_\alpha = F_L^{-1}(\alpha; \mu_L, \sigma_L) = q_\alpha^L$$

where $F_L^{-1}(\cdot; \mu_L, \sigma_L)$ is the quantile function for the normal distribution with mean $\mu_L$ and sd $\sigma_L$.

Note: $F_L^{-1}(\cdot; \mu_L, \sigma_L)$ does not have a closed form solution by can be easily computed numerically in software (e.g. `qnorm()` in R).

Example: R Calculations

```r
> mu = 10
> sigma = 100
> alpha = 0.95
> VaR.alpha = qnorm(alpha, mu, sigma)
> VaR.alpha
[1] 174.4854
```
Result: If $L_{t+1} \sim N(\mu_L, \sigma_L^2)$ then

$$\text{Var}_\alpha = q_{\alpha}^L = \mu_L + \sigma_L \times q_{\alpha}^Z$$

where $q_{\alpha}^Z$ is the $\alpha$-quantile of the standard normal distribution defined by

$$F_Z^{-1}(\alpha) = \Phi^{-1}(\alpha) = q_{\alpha}^Z \text{ s.t. } \Phi(q_{\alpha}^Z) = \alpha$$

where

$$F_Z(x) = \Phi(x) = \Pr(Z \leq x) \text{ and } Z \sim N(0, 1)$$

The proof is easy:

$$\Pr \left( L_{t+1} \geq q_{\alpha}^L \right) = \Pr \left( L_{t+1} \geq \mu_L + \sigma_L \times q_{\alpha}^Z \right)$$

$$= \Pr \left( \frac{L_{t+1} - \mu_L}{\sigma_L} \geq q_{\alpha}^Z \right)$$

$$= \Pr \left( Z \geq q_{\alpha}^Z \right) = \Phi(q_{\alpha}^Z) = \alpha$$
Example: R calculations

\begin{verbatim}
> VaR.alpha = mu + sigma*qnorm(alpha,0,1)
> VaR.alpha
[1] 174.4854
\end{verbatim}

**Expected Shortfall**

Let $F_L$ denote the distribution of losses $L_{t+1}$ on some asset over a given holding period and assume that $F_L$ is continuous.

**Definition 4 (Expected Shortfall, ES).** The expected shortfall at confidence level $\alpha$ is the expected loss conditional on losses being greater than $VaR_\alpha$:

$$ES_\alpha = E[L_{t+1}|L_{t+1} \geq VaR_\alpha]$$

In other words, ES is the expected loss in the upper tail of the loss distribution.

Remark: If $F_L$ is not continuous then $ES_\alpha$ is defined as

$$ES_\alpha = \frac{1}{1 - \alpha} \int_0^1 VaR_u du$$

which is the average of $VaR_u$ over all $u$ that are greater than or equal to $\alpha \in (0,1)$.
Note: To compute $ES_\alpha = E[\mathcal{L}_{t+1}|\mathcal{L}_{t+1} \geq VaR_\alpha]$, you have to compute the mean of the truncated loss distribution

$$ES_\alpha = \frac{\int_{VaR_\alpha}^{\infty} l \times f_L(l)dl}{1 - \alpha}$$

Remarks

• If $\alpha = p$ is the loss probability then

$$ES_p = E[\mathcal{L}_{t+1}|\mathcal{L}_{t+1} \geq VaR_p] = \frac{\int_{VaR_p}^{\infty} l \times f_L(l)dl}{p}$$

• In terms of profits,

$$ES_\alpha = -E[\mathcal{P}_{t+1}|\mathcal{P}_{t+1} \leq -VaR_\alpha]$$

• In terms of returns

$$ES_\alpha = -V_t \times E[\mathcal{-R}_{t+1}| -R_{t+1} \leq q_\alpha]$$
Example: Calculating ES when losses/returns follow a normal distribution

Let $L_{t+1} \sim \mathcal{N}(\mu_L, \sigma_L^2)$ where $\mu_L$ and $\sigma_L$ are known. For confidence level $\alpha$

$$ES_\alpha = E[L_{t+1} | L_{t+1} \geq VaR_\alpha]$$

$$= \text{mean of truncated normal distribution}$$

$$= \mu_L + \sigma_L \times \frac{\phi(q_\alpha^Z)}{1 - \alpha}$$

where $\phi(z) = f_Z(z) = \text{pdf of } Z \sim \mathcal{N}(0, 1)$.

R Calculations

```
\texttt{> mu = 10}
\texttt{> sigma = 100}
\texttt{> alpha = 0.95}
\texttt{> q.alpha.z = qnorm(alpha)}
\texttt{> ES.alpha = mu + sigma*(dnorm(q.alpha.z)/(1-alpha))}
\texttt{> ES.alpha}
\texttt{[[1] 216.2713]}
```
Coherence

Artzner et al. (1999), “Coherent Measures of Risk,” Mathematical Finance, study the properties a risk measure should have in order to be considered a sensible and useful risk measure. They identify four axioms that risk measures should ideally adhere to. A risk measure that satisfies all four axioms is termed coherent.

In what follows, let $RM(\cdot)$ denote a risk measure which could be volatility, VaR or ES.

**Definition 5 (Coherent risk measure).** Consider two random variables $X$ and $Y$ representing asset losses. A function $RM(\cdot): X, Y \rightarrow \mathbb{R}$ is called a coherent risk measure if it satisfies for $X, Y$ and a constant $c$

1. **Monotonicity**

   $X, Y \in V \subset \mathbb{R}, X \geq Y \Rightarrow RM(X) \geq RM(Y)$

   If the loss of $X$ always exceeds the loss $Y$, the risk of $X$ should always exceed the risk of $Y$.

2. **Subadditivity**

   $X, Y, X + Y \in V \Rightarrow RM(X + Y) \leq RM(X) + RM(Y)$

   The risk to the portfolios of $X$ and $Y$ cannot be worse than the sum of the two individual risks - a manifestation of the diversification principle.
3. **Positive homogeneity**

\[ X \in V, c > 0 \Rightarrow RM(cX) = cRM(X) \]

*For example, if the asset value doubles \((c = 2)\) then the risk doubles*

4. **Translation invariance**

\[ X \in V, c \in \mathbb{R} \Rightarrow RM(X + c) = RM(X) + c \]

*For example, adding \(c < 0\) to the loss is like adding cash, which acts as insurance, so the risk of \(X + c\) is less than the risk of \(X\) by the amount of cash, \(c\).*

**Remarks**

1. FRF define coherence using \(X, Y\) representing profits. This alters translation invariance to \(X \in V, c \in \mathbb{R} \Rightarrow RM(X + c) = RM(X) - c\)

2. Positive homogeneity is often violated in practice for large \(c\)

3. It can be shown that ES is a coherent risk measure

4. VaR does not always satisfy subadditivity so is not in general coherent
   
   - This is undesirable because it means that you cannot bound aggregate risk by the weighted sum of individual VaR values
Example: Volatility is subadditive

(do on white board)

Examples: VaR is not subadditive - Next homework assignment
When does VaR violate subadditivity?

- When the tails of assets are super fat!

- When assets are subject to occasional very large returns
  - Exchange rates in countries that peg currency but are subject to occasional large devaluations
  - Electricity prices subject to occasional large price swings
  - Defaultable bonds when most of the time the bonds deliver a steady positive return but may occasionally default

- Protection seller portfolios - portfolios that earn a small positive amount with high probability but suffer large losses with small probabilities (carry trades, short options, insurance contracts)