AMATH 546/ECON 589
Risk Measures

Eric Zivot

April 3, 2013
Outline

• Profit, Loss and Return Distributions

• Risk measures

• Risk measure properties

• Portfolio risk measures and risk budgeting
Reading

- FRF chapter 4
- QRM chapter 2, sections 1 and 2; chapter 6
- FMUND chapter 8
- SADFE chapter 19
Profits and Losses

- $V_t =$ value (price) of an asset at time $t$ (assumed known) measured in $\$

- $V_{t+1} =$ value (price) of an asset at time $t+1$ (typically unknown) measured in $\$

- $\Pi_{t+1} = V_{t+1} - V_t =$ profit measured in $\$ over the holding period

- $L_{t+1} = -\Pi_{t+1} =$ loss measured in $\$ over the holding period
Remarks

- $\Pi_{t+1} > 0$ (positive profit) $\implies L_{t+1} < 0$ (negative loss)

- $\Pi_{t+1} < 0$ (negative profit) $\implies L_{t+1} > 0$ (positive loss)

- Risk measures are typically defined in terms of losses. Hence a large positive value for a risk measure indicates a large positive loss.
Returns

- $R_{t+1} = \frac{V_{t+1} - V_t}{V_t} = $ simple return between times $t$ and $t + 1$

- $\Pi_{t+1} = V_t R_{t+1} = V_{t+1} - V_t$

- $L_{t+1} = -V_t R_{t+1}$
Profit, Loss and Return Distributions

- \( \Pi_{t+1}, L_{t+1} \) and \( R_{t+1} \) are random variables because at time \( t \) the future value \( V_{t+1} \) is not known.

- For the random variable \( X = \Pi_{t+1}, L_{t+1} \) and \( R_{t+1} \), let \( F_X \) denote the CDF and \( f_X \) denote the pdf.

- The distributions of \( \Pi_{t+1}, L_{t+1} \) and \( R_{t+1} \) are obviously linked.

- Assume (unless otherwise specified) that \( F_X \) and \( f_X \) are are known and are continuous functions.
Risk Measurement

Goal: Make risk comparisons across a variety of assets to aid decision making of some sort.

- Determination of risk capital and capital adequacy
- Management tool
- Insurance premiums

**Definition 1** *(Risk measure)* Mathematical method for capturing risk

**Definition 2** *(Risk measurement)* A number that captures risk. It is obtained by applying data to a risk measure
Three Most Common Risk Measures

1. Volatility (vol or $\sigma$)

2. Value-at-Risk (VaR)

3. Expected Shortfall (ES)
   • aka Expected Tail Loss (ETL), conditional VaR (cVaR)
Volatility

Profit/Loss volatility

\[ \sigma_\Pi = \left( E[(\Pi_{t+1} - \mu_\Pi)^2] \right)^{1/2} = \sigma_L = \left( E[(L_{t+1} - \mu_L)^2] \right)^{1/2} \]

Return Volatility

\[ \sigma_R = \left( E[(R_{t+1} - \mu_R)^2] \right)^{1/2} \]

Relationship between \( \sigma_L \) and \( \sigma_R \)

\[ L_{t+1} = V_t R_{t+1} \Rightarrow \sigma_L = V_t \sigma_R \]
Remarks

- Volatility measures the size of a typical deviation from the mean loss or return.

- Volatility is a symmetric risk measure - does not focus on downside risk.

- Volatility is an appropriate risk measure if losses or returns have a normal distribution.

- Volatility might not exist (i.e., might not be a finite number)
Value-at-Risk

Let $F_L$ denote the distribution of losses $L_{t+1}$ on some asset over a given holding period.

**Definition 3** (Value-at-Risk, QRM) Given some confidence level $\alpha \in (0, 1)$. The VaR on our asset at the confidence level $\alpha$ is given by the smallest number $l$ such that the probability that the loss $L_{t+1}$ exceeds $l$ is no larger than $(1 - \alpha)$. Formally,

$$VaR_\alpha = \inf \{l \in \mathbb{R} : \Pr(L_{t+1} > l) \leq 1 - \alpha\}$$

If $F_L$ is continuous then $VaR_\alpha$ can be implicitly defined using

$$\Pr(L_{t+1} \geq VaR_\alpha) = 1 - \alpha$$

or

$$F_L(VaR_\alpha) = \Pr(L_{t+1} \leq VaR_\alpha) = \alpha$$
Remarks

- $VaR_\alpha$ is the upper $\alpha$-quantile of the loss distribution $F_L$. If $F_L$ is continuous then $VaR_\alpha$ can be conveniently computed using the quantile function $F_L^{-1}$

$$VaR_\alpha = F_L^{-1}(\alpha) = q^L_\alpha$$

- Typically $\alpha = 0.90$, 0.95 or 0.99. If $\alpha = 0.95$ then with 95% confidence we could lose $VaR_{0.95}$ or more over the holding period.

- $VaR_\alpha$ is a lower bound on the possible losses that might occur with confidence level $\alpha$. It says nothing about the magnitude of loses beyond $VaR_\alpha$. 
Alternative Definitions of VaR

Unfortunately, there is no universally accepted definition of $VaR_\alpha$

- Some authors define $\alpha$ as the probability of loss. In this case, for continuous $F_L$, we have

$$\Pr(L_{t+1} \geq VaR_\alpha) = \alpha \text{ so that } VaR_\alpha = F_L^{-1}(1 - \alpha) = q_{1-\alpha}^L$$

For example, if $\alpha = 0.05$, then with 5% probability we could lose $VaR_{0.05}$ or more over the holding period.
• Some authors (e.g. FRF) define VaR using the distribution of profits. Since $\Pi_{t+1} = -L_{t+1}$ we have (using $\alpha$ as confidence level)

$$ \Pr(L_{t+1} \geq VaR_\alpha) = 1 - \alpha $$
$$ = \Pr(-\Pi_{t+1} \geq VaR_\alpha) = 1 - \alpha $$
$$ = \Pr(\Pi_{t+1} \leq -VaR_\alpha) = 1 - \alpha $$

Following FRF, if $VaR$ is defined using probability of loss then use $\alpha = p$ to denote loss probability and write $VaR_p$. Then

$$ \Pr(\Pi_{t+1} \leq -VaR_p) = p $$
• Some authors define $VaR$ using the distribution of returns. Since $L_{t+1} = -V_t R_{t+1}$ we have (using $\alpha$ as confidence level)

$$
\Pr(L_{t+1} \geq VaR_\alpha) = 1 - \alpha
$$

$$
= \Pr(-V_t R_{t+1} \geq VaR_\alpha) = 1 - \alpha
$$

$$
= \Pr(-R_{t+1} \geq \frac{VaR_\alpha}{V_t}) = 1 - \alpha
$$

Here, $\frac{VaR_\alpha}{V_t} = q_\alpha^{-R}$ is the upper $\alpha$-quantile of the (negative) returns.
Example: Calculating VaR when losses/returns follow a normal distribution

Let $L_{t+1} \sim N(\mu_L, \sigma_L^2)$ where $\mu_L$ and $\sigma_L$ are known. The pdf and CDF are given by

$$f_L(l; \mu_L, \sigma_L) = \frac{1}{\sqrt{2\pi} \sigma_L^2} e^{-\frac{1}{2} \left( \frac{l - \mu_L}{\sigma_L} \right)^2}$$

$$F_L(l; \mu_L, \sigma_L) = \Pr(L_{t+1} \leq l) = \int_{-\infty}^{l} f_L(x; \mu_L, \sigma_L)dx$$

Then, for a given confidence level $\alpha \in (0, 1)$

$$VaR_\alpha = F_{L}^{-1}(\alpha; \mu_L, \sigma_L) = q_{\alpha}^L$$

where $F_{L}^{-1}(\cdot; \mu_L, \sigma_L)$ is the quantile function for the normal distribution with mean $\mu_L$ and sd $\sigma_L$.

Note: $F_{L}^{-1}(\cdot; \mu_L, \sigma_L)$ does not have a closed form solution but can be easily computed numerically in software (e.g. `qnorm()` in R).
Example: R Calculations

> mu = 10
> sigma = 100
> alpha = 0.95
> VaR.alpha = qnorm(alpha, mu, sigma)
> VaR.alpha
[1] 174.4854
Result: If $L_{t+1} \sim N(\mu_L, \sigma_L^2)$ then

$$\text{VaR}_\alpha = q^L_\alpha = \mu_L + \sigma_L \times q^Z_\alpha$$

where $q^Z_\alpha$ is the $\alpha$-quantile of the standard normal distribution defined by

$$F^{-1}_Z(\alpha) = \Phi^{-1}(\alpha) = q^Z_\alpha \text{ s.t. } \Phi(q^Z_\alpha) = \alpha$$

where

$$F_Z(x) = \Phi(x) = \Pr(Z \leq x) \text{ and } Z \sim N(0, 1)$$
The proof is easy:

\[
\Pr\left( L_{t+1} \geq q^L_{\alpha} \right) = \Pr\left( L_{t+1} \geq \mu_L + \sigma_L \times q^Z_{\alpha} \right) \\
= \Pr\left( \frac{L_{t+1} - \mu_L}{\sigma_L} \geq q^Z_{\alpha} \right) \\
= \Pr\left( Z \geq q^Z_{\alpha} \right) = \Phi(q^Z_{\alpha}) = \alpha
\]
Example: R calculations

> VaR.alpha = mu + sigma*qnorm(alpha,0,1)
> VaR.alpha
[1] 174.4854
Expected Shortfall

Let $F_L$ denote the distribution of losses $L_{t+1}$ on some asset over a given holding period and assume that $F_L$ is continuous.

**Definition 4** (Expected Shortfall, ES). The expected shortfall at confidence level $\alpha$ is the expected loss conditional on losses being greater than $VaR_\alpha$:

$$ES_\alpha = E[L_{t+1}|L_{t+1} \geq VaR_\alpha]$$

In other words, ES is the expected loss in the upper tail of the loss distribution.

Remark: If $F_L$ is not continuous then $ES_\alpha$ is defined as

$$ES_\alpha = \frac{1}{1 - \alpha} \int_{\alpha}^{1} VaR_u du$$

which is the average of $VaR_u$ over all $u$ that are greater than or equal to $\alpha \in (0, 1)$.
Note: To compute $ES_\alpha = E[L_{t+1}|L_{t+1} \geq VaR_\alpha]$, you have to compute the mean of the *truncated* loss distribution

$$ES_\alpha = E[L_{t+1}|L_{t+1} \geq VaR_\alpha] = \frac{\int_{VaR_\alpha}^{\infty} l \times f_L(l)dl}{1 - \alpha}$$
Remarks

• If $\alpha = p$ is the loss probability then

$$ES_p = E[L_{t+1}|L_{t+1} \geq VaR_p] = \frac{\int_{VaR_p}^{\infty} l \times f_L(l)dl}{p}$$

• In terms of profits,

$$ES_\alpha = -E[\Pi_{t+1}|\Pi_{t+1} \leq -VaR_\alpha]$$

• In terms of returns

$$ES_\alpha = -V_t \times E[-R_{t+1}|-R_{t+1} \leq q_\alpha^{-R}]$$
Example: Calculating ES when losses/returns follow a normal distribution

Let $L_{t+1} \sim N(\mu_L, \sigma_L^2)$ where $\mu_L$ and $\sigma_L$ are known. For confidence level $\alpha$

$$ES_\alpha = E[L_{t+1} | L_{t+1} \geq VaR_\alpha]$$

= mean of truncated normal distribution

= $\mu_L + \sigma_L \times \frac{\phi(q_{Z\alpha})}{1 - \alpha}$

where $\phi(z) = f_Z(z) = \text{pdf of } Z \sim N(0, 1)$. 

R Calculations

> mu = 10
> sigma = 100
> alpha = 0.95
> q.alpha.z = qnorm(alpha)
> ES.alpha = mu + sigma*(dnorm(q.alpha.z)/(1-alpha))
> ES.alpha

[1] 216.2713
Coherence

Artzner et al. (1999), “Coherent Measures of Risk,” Mathematical Finance, study the properties a risk measure should have in order to be considered a sensible and useful risk measure. They identify four axioms that risk measures should ideally adhere to. A risk measure that satisfies all four axioms is termed *coherent*.

In what follows, let $RM(\cdot)$ denote a risk measure which could be volatility, VaR or ES.
Definition 5 (Coherent risk measure). Consider two random variables $X$ and $Y$ representing asset losses. A function $RM(\cdot) : X, Y \rightarrow \mathbb{R}$ is called a coherent risk measure if it satisfies for $X, Y$ and a constant $c$

1. **Monotonicity**

$$X, Y \in V \subset \mathbb{R}, X \geq Y \Rightarrow RM(X) \geq RM(Y)$$

*If the loss of $X$ always exceeds the loss $Y$, the risk of $X$ should always exceed the risk of $Y*."

2. **Subadditivity**

$$X, Y, X + Y \in V \Rightarrow RM(X + Y) \leq RM(X) + RM(Y)$$

*The risk to the portfolios of $X$ and $Y$ cannot be worse than the sum of the two individual risks - a manifestation of the diversification principle.**
3. **Positive homogeneity**

\[ X \in V, c > 0 \Rightarrow RM(cX) = cRM(X) \]

*For example, if the asset value doubles \((c = 2)\) then the risk doubles*

4. **Translation invariance**

\[ X \in V, c \in \mathbb{R} \Rightarrow RM(X + c) = RM(X) + c \]

*For example, adding \(c < 0\) to the loss is like adding cash, which acts as insurance, so the risk of \(X + c\) is less than the risk of \(X\) by the amount of cash, \(c\).*
Remarks

1. FRF define coherence using $X, Y$ representing profits. This alters translation invariance to $X \in V, c \in \mathbb{R} \Rightarrow RM(X + c) = RM(X) - c$

2. Positive homogeneity is often violated in practice for large $c$ due to liquidity effects

3. It can be shown that ES is a coherent risk measure but that volatility and VaR are not
4. VaR does not always satisfy subadditivity so is not in general coherent

- This is undesirable because it means that you cannot bound aggregate risk by the weighted sum of individual VaR values.
Example: Volatility is subadditive

Consider a portfolio of 2 assets X and Y with returns $R_x$ and $R_y$ and portfolio weights $w_x > 0$ and $w_y > 0$ s.t. $w_x + w_y = 1$. Then

$$\sigma_p = \left( w_x^2 \sigma_x^2 + w_y^2 \sigma_y^2 + 2w_x w_y \sigma_{xy} \right)^{1/2} \leq w_x \sigma_x + w_y \sigma_y$$

portfolio vol $\leq$ weighted average of standalone vol

Proof:

$$\sigma_p^2 = w_x^2 \sigma_x^2 + w_y^2 \sigma_y^2 + 2w_x w_y \sigma_{xy} = w_x^2 \sigma_x^2 + w_y^2 \sigma_y^2 + 2w_x w_y \sigma_x \sigma_y \rho_{xy}$$

Now add and subtract $2w_x w_y \sigma_x \sigma_y$

$$\sigma_p^2 = \left( w_x^2 \sigma_x^2 + w_y^2 \sigma_y^2 + 2w_x w_y \sigma_{xy} \right) - 2w_x w_y \sigma_x \sigma_y + 2w_x w_y \sigma_x \sigma_y \rho_{xy}$$

$$= (w_x \sigma_x + w_y \sigma_y)^2 - 2w_x w_y \sigma_x \sigma_y (1 - \rho_{xy})$$
Because

$$2w_xw_y\sigma_x\sigma_y(1 - \rho_{xy}) > 0$$

It follows that

$$\sigma_p = \left[ (w_x\sigma_x + w_y\sigma_y)^2 - 2w_xw_y\sigma_x\sigma_y(1 - \rho_{xy}) \right]^{1/2} \leq w_x\sigma_x + w_y\sigma_y$$

and so portfolio volatility is sub-additive.

However, portfolio volatility is not translation invariant (or monotonic) so it is not coherent; e.g.,

$$var(R + c) = var(R)$$ for any constant \(c\)
Examples: VaR is not subadditive

Consider two traders in the same firm:

1. Short a call option that is deep out of the money, with a 4% chance of losing money

2. Short a put option that is deep out of the money, with a 4% chance of losing money

Neither position has a VaR risk at the 5% level. Yet together, the joint position can have a VaR risk at the 5% level.
To see this, suppose the two positions are independent. Then

\[
\Pr(\text{at least one loses money}) = 1 - \Pr(\text{neither loses money})
\]

\[
= 1 - (0.96)^2 = 0.078 > 5%
\]

Hence, 5% VaR is not sub-additive.
When does VaR violate subadditivity?

- When the tails of assets are super fat!
- When assets are subject to occasional very large returns
  - Exchange rates in countries that peg currency but are subject to occasional large devaluations
  - Electricity prices subject to occasional large price swings
  - Defaultable bonds when most of the time the bonds deliver a steady positive return but may occasionally default
– Protection seller portfolios - portfolios that earn a small positive amount with high probability but suffer large losses with small probabilities (carry trades, short options, insurance contracts)