AMATH 546/ECON 589
Risk Measures

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Outline

- Profit, Loss and Return Distributions
- Risk measures
- Risk measure properties
- Portfolio risk measures and risk budgeting

Reading

- FRF chapter 4
- QRM chapter 2, sections 1 and 2; chapter 6
- FMUND chapter 8
- SADFE chapter 19

Profits and Losses

- V_t = value (price) of an asset at time t (assumed known) measured in \$

- V_{t+1} = value (price) of an asset at time $t+1$ (typically unknown) measured in \$

- $\Pi_{t+1} = V_{t+1} - V_t$ = profit measured in \$ over the holding period

- $L_{t+1} = -\Pi_{t+1}$ = loss measured in \$ over the holding period

Remarks

- $\Pi_{t+1} > 0$ (positive profit) $\Longrightarrow L_{t+1} < 0$ (negative loss)

- $\Pi_{t+1} < 0$ (negative profit) $\Longrightarrow L_{t+1} > 0$ (positive loss)

- Risk measures are typically defined in terms of losses. Hence a large positive value for a risk measure indicates a large positive loss.

Returns

- $R_{t+1} = \frac{V_{t+1} - V_t}{V_t}$ = simple return between times t and $t+1$

- $\Pi_{t+1} = V_t R_{t+1} = V_{t+1} - V_t$

- $L_{t+1} = -V_t R_{t+1}$

Profit, Loss and Return Distributions

- Π_{t+1}, L_{t+1} and R_{t+1} are random variables because at time t the future value V_{t+1} is not known.

- For the random variable $X = \Pi_{t+1}, L_{t+1}$ and $R_{t=1}$, let F_X denote the CDF and f_X denote the pdf

- The distributions of Π_{t+1}, L_{t+1} and R_{t+1} are obviously linked

- Assume (unless otherwise specified) that F_X and f_X are are known and are continuous functions

Risk Measurement

Goal: Make risk comparisons across a variety of assets to aid decision making of some sort.

- Determination of risk capital and capital adequacy

- Management tool

- Insurance premiums

Definition 1 *(Risk measure) Mathematical method for capturing risk*

Definition 2 *(Risk measurement) A number that captures risk. It is obtained by applying data to a risk measure*

Three Most Common Risk Measures

1. Volatility (vol or σ)

2. Value-at-Risk (VaR)

3. Expected Shortfall (ES)

    - aka Expected Tail Loss (ETL), conditional VaR (cVaR)

Volatility

Profit/Loss volatility

$$\sigma_{\Pi} = \left(E[(\Pi_{t+1} - \mu_{\Pi})^2]\right)^{1/2} = \sigma_L = \left(E[(L_{t+1} - \mu_L)^2]\right)^{1/2}$$

Return Volatility

$$\sigma_R = \left(E[(R_{t+1} - \mu_R)^2]\right)^{1/2}$$

Relationship between σ_L and σ_R

$$L_{t+1} = V_t R_{t+1} \Rightarrow \sigma_L = V_t \sigma_R$$

Remarks

- volatility measures the size of a typical deviation from the mean loss or return.

- volatility is a symmetric risk measure - does not focus on downside risk.

- volatility is an appropriate risk measure if losses or returns have a normal distribution.

- volatility might not exist (i.e., might not be a finite number)

Value-at-Risk

Let F_L denote the distribution of losses L_{t+1} on some asset over a given holding period.

Definition 3 *(Value-at-Risk, QRM) Given some confidence level* $\alpha \in (0,1)$*. The VaR on our asset at the confidence level* α *is given by the smallest number* l *such that the probability that the loss* L_{t+1} *exceeds* l *is no larger than* $(1-\alpha)$*. Formally,*

$$VaR_\alpha = \inf \{l \in \mathbb{R} : \Pr(L_{t+1} > l) \leq 1 - \alpha\}$$

If F_L *is continuous then* VaR_α *can be implicitly defined using*

$$\Pr(L_{t+1} \geq VaR_\alpha) = 1 - \alpha$$

or

$$F_L(VaR_\alpha) = \Pr(L_{t+1} \leq VaR_\alpha) = \alpha$$

Remarks

- VaR_α is the upper α-quantile of the loss distribution F_L. If F_L is continuous then VaR_α can be conveniently computed using the quantile function F_L^{-1}

$$VaR_\alpha = F_L^{-1}(\alpha) = q_\alpha^L$$

- Typically $\alpha = 0.90$, 0.95 or 0.99. If $\alpha = 0.95$ then with 95% confidence we could lose $VaR_{0.95}$ or more over the holding period.

- VaR_α is a *lower bound* on the possible losses that might occur with confidence level α. It says nothing about the magnitude of loses beyond VaR_α.

Alternative Definitions of VaR

Unfortunately, there is no universally accepted definition of VaR_α

- Some authors define α as the *probability of loss*. In this case, for continuous F_L, we have

$$\Pr(L_{t+1} \geq VaR_\alpha) = \alpha \text{ so that } VaR_\alpha = F_L^{-1}(1-\alpha) = q_{1-\alpha}^L$$

For example, if $\alpha = 0.05$, then with 5% probability we could lose $VaR_{0.05}$ or more over the holding period.

- Some authors (e.g. FRF) define VaR using the distribution of profits. Since $\Pi_{t+1} = -L_{t+1}$ we have (using α as confidence level)

$$\begin{aligned} \Pr(L_{t+1} &\geq VaR_\alpha) = 1 - \alpha \\ &= \Pr(-\Pi_{t+1} \geq VaR_\alpha) = 1 - \alpha \\ &= \Pr(\Pi_{t+1} \leq -VaR_\alpha) = 1 - \alpha \end{aligned}$$

Following FRF, if VaR is defined using probability of loss then use $\alpha = p$ to denote loss probability and write VaR_p. Then

$$\Pr(\Pi_{t+1} \leq -VaR_p) = p$$

- Some authors define VaR using the distribution of returns. Since $L_{t+1} = -V_t R_{t+1}$ we have (using α as confidence level)

$$\begin{aligned} \Pr(L_{t+1} &\geq VaR_\alpha) = 1-\alpha \\ &= \Pr(-V_t R_{t+1} \geq VaR_\alpha) = 1-\alpha \\ &= \Pr(-R_{t+1} \geq \frac{VaR_\alpha}{V_t}) = 1-\alpha \end{aligned}$$

Here, $\frac{VaR_\alpha}{V_t} = q_\alpha^{-R}$ is the upper α-quantile of the (negative) returns.

Example: Calculating VaR when losses/returns follow a normal distribution

Let $L_{t+1} \sim N(\mu_L, \sigma_L^2)$ where μ_L and σ_L are known. The pdf and CDF are given by

$$\begin{aligned} f_L(l; \mu_L, \sigma_L) &= \frac{1}{\sqrt{2\pi\sigma_L^2}} e^{-\frac{1}{2}\left(\frac{l-\mu_L}{\sigma_L}\right)^2} \\ F_L(l; \mu_L, \sigma_L) &= \Pr(L_{t+1} \leq l) = \int_{-\infty}^{l} f_L(x; \mu_L, \sigma_L) dx \end{aligned}$$

Then, for a given confidence level $\alpha \in (0, 1)$

$$VaR_\alpha = F_L^{-1}(\alpha; \mu_L, \sigma_L) = q_\alpha^L$$

where $F_L^{-1}(\cdot; \mu_L, \sigma_L)$ is the quantile function for the normal distribution with mean μ_L and sd σ_L.

Note: $F_L^{-1}(\cdot; \mu_L, \sigma_L)$ does not have a closed form solution by can be easily computed numerically in software (e.g. `qnorm()` in R).

Example: R Calculations

```
> mu = 10
> sigma = 100
> alpha = 0.95
> VaR.alpha = qnorm(alpha, mu, sigma)
> VaR.alpha
[1] 174.4854
```

Result: If $L_{t+1} \sim N(\mu_L, \sigma_L^2)$ then

$$VaR_\alpha = q_\alpha^L = \mu_L + \sigma_L \times q_\alpha^Z$$

where q_α^Z is the α-quantile of the standard normal distribution defined by

$$F_Z^{-1}(\alpha) = \Phi^{-1}(\alpha) = q_\alpha^Z \text{ s.t. } \Phi(q_\alpha^Z) = \alpha$$

where

$$F_Z(x) = \Phi(x) = \Pr(Z \leq x) \text{ and } Z \sim N(0, 1)$$

The proof is easy:

$$\begin{aligned}
\Pr\left(L_{t+1} \geq q_\alpha^L\right) &= \Pr\left(L_{t+1} \geq \mu_L + \sigma_L \times q_\alpha^Z\right) \\
&= \Pr\left(\frac{L_{t+1} - \mu_L}{\sigma_L} \geq q_\alpha^Z\right) \\
&= \Pr\left(Z \geq q_\alpha^Z\right) = \Phi(q_\alpha^Z) = \alpha
\end{aligned}$$

Example: R calculations

```
> VaR.alpha = mu + sigma*qnorm(alpha,0,1)
> VaR.alpha
[1] 174.4854
```

Expected Shortfall

Let F_L denote the distribution of losses L_{t+1} on some asset over a given holding period and assume that F_L is continuous.

Definition 4 *(Expected Shortfall, ES). The expected shortfall at confidence level α is the expected loss conditional on losses being greater than VaR_α :*

$$ES_\alpha = E[L_{t+1}|L_{t+1} \geq VaR_\alpha]$$

In other words, ES is the expected loss in the upper tail of the loss distribution.

Remark: If F_L is not continuous then ES_α is defined as

$$ES_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 VaR_u du$$

which is the average of VaR_u over all u that are greater than or equal to $\alpha \in (0, 1)$

Note: To compute $ES_\alpha = E[L_{t+1}|L_{t+1} \geq VaR_\alpha]$, you have to compute the mean of the *truncated* loss distribution

$$\begin{aligned} ES_\alpha &= E[L_{t+1}|L_{t+1} \geq VaR_\alpha] \\ &= \frac{\int_{VaR_\alpha}^{\infty} l \times f_L(l)dl}{1-\alpha} \end{aligned}$$

Remarks

- If $\alpha = p$ is the loss probability then

$$ES_p = E[L_{t+1} | L_{t+1} \geq VaR_p] = \frac{\int_{VaR_p}^{\infty} l \times f_L(l) dl}{p}$$

- In terms of profits,

$$ES_\alpha = -E[\Pi_{t+1} | \Pi_{t+1} \leq -VaR_\alpha]$$

- In terms of returns

$$ES_\alpha = -V_t \times E[-R_{t+1} | - R_{t+1} \leq q_\alpha^{-R}]$$

Example: Calculating ES when losses/returns follow a normal distribution

Let $L_{t+1} \sim N(\mu_L, \sigma_L^2)$ where μ_L and σ_L are known. For confidence level α

$$
\begin{aligned}
ES_\alpha &= E[L_{t+1} | L_{t+1} \geq VaR_\alpha] \\
&= \text{ mean of truncated normal distribution} \\
&= \mu_L + \sigma_L \times \frac{\phi(q_\alpha^Z)}{1-\alpha}
\end{aligned}
$$

where $\phi(z) = f_Z(z) =$ pdf of $Z \sim N(0,1)$.

R Calculations

```
> mu = 10
> sigma = 100
> alpha = 0.95
> q.alpha.z = qnorm(alpha)
> ES.alpha = mu + sigma*(dnorm(q.alpha.z)/(1-alpha))
> ES.alpha
[1] 216.2713
```

Coherence

Artzner et al. (1999), "Coherent Measures of Risk," Mathematical Finance, study the properties a risk measure should have in order to be considered a sensible and useful risk measure. They identify four axioms that risk measures should ideally adhere to. A risk measure that satisfies all four axioms is termed *coherent*.

In what follows, let $RM(\cdot)$ denote a risk measure which could be volatility, VaR or ES.

Definition 5 *(Coherent risk measure). Consider two random variables X and Y representing asset losses. A function $RM(\cdot) : X, Y \to \mathbb{R}$ is called a coherent risk measure if it satisfies for X, Y and a constant c*

1. *Monotonicity*

$$X, Y \in V \subset \mathbb{R}, X \geq Y \Rightarrow RM(X) \geq RM(Y)$$

   If the loss of X always exceeds the loss Y, the risk of X should always exceed the risk of Y.

2. *Subaddivitity*

$$X, Y, X + Y \in V \Rightarrow RM(X + Y) \leq RM(X) + RM(Y)$$

   The risk to the portfolios of X and Y cannot be worse than the sum of the two individual risks - a manifestation of the diversification principle.

3. *Positive homogeneity*

$$X \in V, c > 0 \Rightarrow RM(cX) = cRM(X)$$

For example, if the asset value doubles $(c = 2)$ *then the risk doubles*

4. *Translation invariance*

$$X \in V, c \in \mathbb{R} \Rightarrow RM(X + c) = RM(X) + c$$

For example, adding $c < 0$ *to the loss is like adding cash, which acts as insurance, so the risk of* $X + c$ *is less than the risk of* X *by the amount of cash,* c.

Remarks

1. FRF define coherence using X, Y representing profits. This alters translation invariance to $X \in V, c \in \mathbb{R} \Rightarrow RM(X + c) = RM(X) - c$

2. Positive homogeneity is often violated in practice for large c due to liquidity effects

3. It can be shown that ES is a coherent risk measure but that volatility and VaR are not

4. VaR does not always satisfy subadditivity so is not in general coherent

   - This is undesirable because it means that you cannot bound aggregate risk by the weighted sum of individual VaR values

Example: Volatility is subadditive

Consider a portfolio of 2 assets X and Y with returns R_x and R_y and portfolio weights $w_x > 0$ and $w_y > 0$ s.t. $w_x + w_y = 1$. Then

$$\begin{aligned} \sigma_p &= \left(w_x^2\sigma_x^2 + w_y^2\sigma_y^2 + 2w_x w_y \sigma_{xy}\right)^{1/2} \leq w_x\sigma_x + w_y\sigma_y \\ \text{portfolio vol} &\leq \text{weighted average of standalone vol} \end{aligned}$$

Proof:

$$\sigma_p^2 = w_x^2\sigma_x^2 + w_y^2\sigma_y^2 + 2w_x w_y \sigma_{xy} = w_x^2\sigma_x^2 + w_y^2\sigma_y^2 + 2w_x w_y \sigma_x \sigma_y \rho_{xy}$$

Now add and subtract $2w_x w_y \sigma_x \sigma_y$

$$\begin{aligned} \sigma_p^2 &= \left(w_x^2\sigma_x^2 + w_y^2\sigma_y^2 + 2w_x w_y \sigma_x \sigma_y\right) - 2w_x w_y \sigma_x \sigma_y + 2w_x w_y \sigma_x \sigma_y \rho_{xy} \\ &= (w_x\sigma_x + w_y\sigma_y)^2 - 2w_x w_y \sigma_x \sigma_y (1 - \rho_{xy}) \end{aligned}$$

Because

$$2w_x w_y \sigma_x \sigma_y (1 - \rho_{xy}) > 0$$

It follows that

$$\sigma_p = \left[(w_x \sigma_x + w_y \sigma_y)^2 - 2w_x w_y \sigma_x \sigma_y (1 - \rho_{xy}) \right]^{1/2} \leq w_x \sigma_x + w_y \sigma_y$$

and so portfolio volatility is sub-additive.

However, portfolio volatility is not translation invariant (or monotonic) so it is not coherent; e.g.,

$$var(R + c) = var(R) \text{ for any constant } c$$

Examples: VaR is not subadditive

Consider two traders in the same firm:

1. Short a call option that is deep out of themoney, with a 4% chance of losing money

2. Short a put option that is deep out of the money, with a 4% chance of losing money

Neither position has a VaR risk at the 5% level. Yet together, the joint position can have a VaR risk at the 5% level.

To see this, suppose the two positions are independent. Then

$$
\begin{aligned}
\text{Pr(at least one loses money)} &= 1 - \text{Pr(neither loses money)} \\
&= 1 - (0.96)^2 = 0.078 > 5\%
\end{aligned}
$$

Hence, 5% VaR is not sub-additive.

When does VaR violate subadditivity?

- When the tails of assets are super fat!

- When assets are subject to occasional very large returns
  - Exchange rates in countries that peg currency but are subject to occasional large devaluations
  - Electricity prices subject to occasional large price swings
  - Defaultable bonds when most of the time the bonds deliver a steady positive return but may occasionally default

– Protection seller portfolios - portfolios that earn a small positive amount with high probability but suffer large losses with small probabilities (carry trades, short options, insurance contracts)