Asset Risk Measures

Let $L_1, L_2, \ldots$ are iid random variables representing risks or losses associated with some investment with CDF F

Example 1. Let R_t denote the daily simple return on an asset and let W_0 denote the initial value of the investment. Then the daily dollar return is

$$R_t W_0$$

If $R_t > 0$ then there is a profit and if $R_t < 0$ then there is a loss. If, by convention, losses are reported as positive amounts then the loss distribution F is the the distribution of

$$\begin{aligned} \text{dollar losses} &: \quad L_t = -R_t W_0 \\ \text{percent losses} &: \quad L_t = -R_t \end{aligned}$$

Example 2: Let $r_t = \ln(1 + R_t)$ denote the daily continuously compounded return on an asset and let W_0 denote the initial value of the investment. Then the daily dollar return is

$$W_0 (\exp(r_t) - 1) = R_t W_0$$

The loss distribution F is the distribution of

$$\begin{aligned} \text{dollar loss} \quad &: \quad L_t = -W_0(\exp(r_t) - 1) = -W_0 R_t \\ \text{percent loss} \quad &: \quad L_t = -(\exp(r_t) - 1) = -R_t \end{aligned}$$

Value-at-Risk (VaR). For $0.95 \leq q < 1$, say, VaR_q is the qth quantile of the distribution F

$$VaR_q = F^{-1}(q)$$

where F^{-1} is the inverse of F.

Expected Shortfall (ES). ES_q is the expected loss size, given that VaR_q is exceeded:

$$ES_q = E[L|L > VaR_q]$$

Note: Writing $L = VaR_q + L - VaR_q$, ES_q is related to VaR_q via

$$ES_q = VaR_q + E[L - VaR_q|L > VaR_q]$$

Remark:

For positive losses, q is the probability level associated with the upper quantile of F. Sometime, VaR and ES are stated in terms of the loss loss probability $\alpha = 1 - q$ for $0 < \alpha \leq 0.05$. Then

$$\begin{aligned} VaR_\alpha &= F^{-1}(1-\alpha) \\ ES_\alpha &= E[L|L > VaR_\alpha] \end{aligned}$$

Example: VaR and ES for normal distribution: $L \sim N(\mu, \sigma^2)$

VaR:

$$\begin{aligned}
VaR_q^N &= \mu + \sigma \cdot z_q,\ 0.95 \le q < 1 \\
VaR_\alpha^N &= -\mu - \sigma \cdot z_\alpha,\ 0 < \alpha < 0.05 \\
z_q &= \ q \cdot 100\% \text{ upper quantile for } N(0,1) \\
z_\alpha &= \ \alpha \cdot 100\% \text{ lower quantile for } N(0,1)
\end{aligned}$$

ES:

$$\begin{aligned}
ES_q^N &= \mu + \sigma \cdot \frac{\phi(z)}{1 - \Phi(z)} = \mu + \sigma \cdot \frac{\phi(z_q)}{1 - q} \\
ES_\alpha^N &= -\mu + \sigma \cdot \frac{\phi(z_\alpha)}{\alpha} \\
z &= (VaR_q - \mu)/\sigma
\end{aligned}$$

Nonparametric VaR and ES (Historical Simulation)

$$\begin{aligned}
\{L_1, \ldots, L_T\} &= \text{sample of losses} \\
VaR_q^{HS} &= \hat{F}^{-1}(q) = \text{empirical quantile} \\
ES_q^{HS} &= \frac{1}{(1-q)T} \sum_{t+1}^{T} L_t \times \mathbf{1}\left(L_t > VaR_q^{HS}\right) \\
\mathbf{1}\left(L_t > VaR_q^{HS}\right) &= 1 \text{ if } L_t > VaR_q^{HS} \\
&= 0, \text{ otherwise}
\end{aligned}$$

VaR for location-scale Return Distributions

Assume that L_t can be represented as

$$L_t = \mu + \sigma u_t, \ u_t = (L - \mu)/\sigma$$
$$E[L_t] = \mu, \ \mathsf{var}(L_t) = \sigma^2$$
$$u_t \sim iid \ (0, 1) \text{ with } CDF \ F_u$$

Then

$$VaR_q = F^{-1}(q) = \mu + \sigma \cdot F_u^{-1}(q)$$

$$\begin{aligned}
\text{normal VaR} &: \quad F_u^{-1}(q) = \text{N(0,1) quantile} \\
\text{Student's t VaR} &: \quad F_u^{-1}(q) = \text{Student's t quantile} \\
\text{Cornish-Fisher (modified) VaR} &: \quad F_u^{-1}(q) = \text{Cornish-Fisher quantile} \\
\text{EVT VaR} &: \quad F_u^{-1}(q) = \text{GPD quantile}
\end{aligned}$$

Cornish-Fisher quantile estimate

Idea: Approximate unknown CDF F_u using 2 term Edgeworth expansion around normal CDF $\Phi(\cdot)$ and invert expansion to get quantile estimate:

$$\begin{aligned} F_{u,CF}^{-1}(q) &= z_q + \frac{1}{6}(z_q^2 - 1) \times skew_u + \frac{1}{24}(z_q^3 - 3z_q) \times kurt_u \\ &\quad -\frac{1}{36}(2z_q^3 - 5z_q) \times skew_u \\ z_q &= \Phi^{-1}(q) \end{aligned}$$

Reference:

Boudt, Peterson and Croux (2008) "Estimation and Decomposition of Downside Risk for Portfolios with Nonnormal Returns," *Journal of Risk.*

R package PerformanceAnalytics