AMATH 546/ECON 589
Risk Budgeting

Eric Zivot

April 10, 2012

Outline

- Portfolio Calculations
- Risk Budgeting
- Reverse Optimization and Implied Returns

Portfolio Risk Budgeting

- Additively decompose (slice and dice) portfolio risk measures into asset contributions
- Allow portfolio manager to know sources of asset risk for allocation and hedging purposes
- Allow risk manager to evaluate portfolio from asset risk perspective

Portfolio Calculations

Let $R_1, \ldots, R_n$ denote simple returns on n assets, and let $w_1, \ldots, w_n$ denote portfolio weights such that $\sum_{i=1}^n w_i = 1$.

Portfolio return:

$$
\begin{aligned}
\mathbf{R} &= (R_1, \ldots, R_N),\ \mathbf{w} = (w_1, \ldots, w_n)',\ \mathbf{1} = (1, \ldots, 1)' \\
R_p &= \mathbf{w}'\mathbf{R} = \sum_{i=1}^{N} w_i R_i,\ \mathbf{w}'\mathbf{1} = 1
\end{aligned}
$$

Portfolio mean and variance:

Let $\mathbf{R}$ be a random vector with

$$
\begin{aligned}
E[\mathbf{R}] &= \boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)' \\
var(\mathbf{R}) &= E[(\mathbf{R}-\boldsymbol{\mu})(\mathbf{R}-\boldsymbol{\mu})'] = \mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1n} & \sigma_{2n} & \cdots & \sigma_n^2 \end{pmatrix}
\end{aligned}
$$

Then

$$
\mu_p = \mathbf{w}'\boldsymbol{\mu},\ \sigma_p^2 = \mathbf{w}'\mathbf{\Sigma}\mathbf{w} \text{ and } \sigma_p = \left(\mathbf{w}'\mathbf{\Sigma}\mathbf{w}\right)^{1/2}
$$

Example: Portfolio risk decomposition for 2 risky asset portfolio

$$\begin{aligned} R_p &= w_1 R_1 + w_2 R_2 \\ \sigma_p^2 &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12} \\ \sigma_p &= \left(w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}\right)^{1/2} \end{aligned}$$

To get an additive decomposition for σ_p^2 write

$$\begin{aligned} \sigma_p^2 &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12} \\ &= \left(w_1^2\sigma_1^2 + w_1w_2\sigma_{12}\right) + \left(w_2^2\sigma_2^2 + w_1w_2\sigma_{12}\right). \end{aligned}$$

Here we can split the covariance contribution $2w_1w_2\sigma_{12}$ to portfolio variance evenly between the two assets and define

$$\begin{aligned} w_1^2\sigma_1^2 + w_1w_2\sigma_{12} &= \text{variance contribution of asset 1} \\ w_2^2\sigma_2^2 + w_1w_2\sigma_{12} &= \text{variance contribution of asset 2} \end{aligned}$$

We can also define an additive decomposition for σ_p

$$
\begin{aligned}
\sigma_p &= \frac{w_1^2\sigma_1^2 + w_1 w_2 \sigma_{12}}{\sigma_p} + \frac{w_2^2\sigma_2^2 + w_1 w_2 \sigma_{12}}{\sigma_p} \\
\frac{w_1^2\sigma_1^2 + w_1 w_2 \sigma_{12}}{\sigma_p} &= \text{sd contribution of asset 1} \\
\frac{w_2^2\sigma_2^2 + w_1 w_2 \sigma_{12}}{\sigma_p} &= \text{sd contribution of asset 2}
\end{aligned}
$$

Euler's Theorem and Risk Decompositions

- When we used σ_p to measure portfolio risk, we were able to easily derive an additive risk decomposition.
- If we measure portfolio risk by VaR or ES it is not so obvious how to define individual asset risk contributions.
- For portfolio risk measures that are homogenous functions of degree one in the portfolio weights, Euler's theorem provides a general method for additively decomposing risk into asset specific contributions.

Homogenous functions and Euler's theorem

First we define a homogenous function of degree one.

Definition 1 *homogenous function of degree one*

Let $f(w_1, \dots, w_n)$ be a continuous and differentiable function of the variables $w_1, \dots, w_n$. f is homogeneous of degree one if for any constant $c > 0$, $f(c \cdot w_1, \dots, c \cdot w_n) = c \cdot f(w_1, \dots, w_n)$.

Note: In matrix notation we have $f(w_1, \dots, w_n) = f(w)$ where

$w = (w_1, \dots, w_n)'$. Then f is homogeneous of degree one if $f(c \cdot w) = c \cdot f(w)$

Examples

Let $f(w_1, w_2) = w_1 + w_2$. Then

$$f(c \cdot w_1, c \cdot w_2) = c \cdot w_1 + c \cdot w_2 = c \cdot (w_1 + w_2) = c \cdot f(w_1, w_2)$$

Let $f(w_1, w_2) = w_1^2 + w_2^2$. Then

$$f(c \cdot w_1, c \cdot w_2) = c^2 w_1^2 + w_2^2 c^2 = c^2 (w_1^2 + w_2^2) \neq c \cdot f(w_1, w_2)$$

Let $f(w_1, w_2) = \sqrt{w_1^2 + w_2^2}$ Then

$$f(c \cdot w_1, c \cdot w_2) = \sqrt{c^2 w_1^2 + c^2 w_2^2} = c\sqrt{(w_1^2 + w_2^2)} = c \cdot f(w_1, w_2)$$

Repeat examples using matrix notation

Define $w = (w_1, w_2)'$ and $\mathbf{1} = (1, 1)'$.

Let $f(w_1, w_2) = w_1 + w_2 = w'\mathbf{1} = \mathbf{f}(w)$. Then

$$f(c \cdot \mathbf{w}) = (c \cdot \mathbf{w})'\,\mathbf{1} = c \cdot (\mathbf{w}'\mathbf{1}) = c \cdot f(\mathbf{w}).$$

Let $f(w_1, w_2) = w_1^2 + w_2^2 = w'w = f(w)$. Then

$$f(c \cdot \mathbf{w}) = (c \cdot \mathbf{w})'(c \cdot \mathbf{w}) = c^2 \cdot \mathbf{w}'\mathbf{w} \neq c \cdot f(\mathbf{w}).$$

Let $f(w_1, w_2) = \sqrt{w_1^2 + w_2^2} = (w'w)^{1/2} = f(w)$. Then

$$f(c \cdot \mathbf{w}) = \Big((c \cdot \mathbf{w})'(c \cdot \mathbf{w})\Big)^{1/2} = c \cdot \Big(\mathbf{w}'\mathbf{w}\Big)^{1/2} = c \cdot f(\mathbf{w}).$$

Consider a portfolio of n assets $w = (w_1, \ldots, w_n)'$ with initial value V_0 and let $\alpha \in (0, 1)$ denote a confidence level

$$\begin{aligned}
\mathbf{R} &= (R_1, \ldots, R_n)',\ \mathbf{w} = (w_1, \ldots, w_n)' \\
E[\mathbf{R}] &= \boldsymbol{\mu},\ \mathrm{cov}(\mathbf{R}) = \boldsymbol{\Sigma},\ \mathbf{R} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})
\end{aligned}$$

Define

$$\begin{aligned}
R_p &= R_p(\mathbf{w}) = \mathbf{w}'\mathbf{R}, \\
\mu_p &= \mu_p(\mathbf{w}) = \mathbf{w}'\boldsymbol{\mu},\ \sigma_p^2 = \sigma_p^2(\mathbf{w}) = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}, \\
\sigma_p &= \sigma_p(\mathbf{w}) = (\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w})^{1/2} \\
q_{1-\alpha}^{R_p} &= q_{1-\alpha}^{R_p}(\mathbf{w}) = \mu_p(\mathbf{w}) + \sigma_p(\mathbf{w}) \times q_{1-\alpha}^{Z} \\
\mathrm{VaR}_\alpha(\mathbf{w}) &= -q_{1-\alpha}^{R_p}(\mathbf{w}) \times V_0 \\
ES_\alpha(w) &= -V_0 \left(\mu_p(\mathbf{w}) + \sigma_p(\mathbf{w}) \times \frac{\phi(q_{1-\alpha}^{Z})}{1-\alpha}\right)
\end{aligned}$$

Result: Portfolio return $R_p(w)$, expected return $\mu_p(w)$, standard deviation $\sigma_p(w)$, normal quantile $q_{1-\alpha}^{R_p}(w)$, and normal VaR $\mathsf{VaR}_\alpha(w)$, and normal ES $ES_\alpha(\mathbf{w})$ are homogenous functions of degree one in the portfolio weight vector w.

Remarks

- Above results for VaR and ES are based on assuming normally distributed returns

- It can be shown that linear homogeneity of VaR and ES holds for any distribution of returns

Let $RM(\mathbf{w})$ denote the risk measures σ, VaR_α and ES_α defined from returns as functions of the portfolio weights $\mathbf{w}$.

Result: $RM(\mathbf{w})$ is a linearly homogenous function of $\mathbf{w}$ for $RM = \sigma, VaR_\alpha$ and ES_α. That is, $RM(c \cdot \mathbf{w}) = c \cdot RM(\mathbf{w})$ for any constant $c \geq 0$

Theorem 2 *Euler's theorem*

Let $f(w_1, \ldots, w_n) = f(w)$ be a continuous, differentiable and homogenous of degree one function of the variables $w = (w_1, \ldots, w_n)'$. Then

$$\begin{aligned} f(\mathbf{w}) &= w_1 \cdot \frac{\partial f(\mathbf{w})}{\partial w_1} + w_2 \cdot \frac{\partial f(\mathbf{w})}{\partial w_2} + \cdots + w_n \cdot \frac{\partial f(\mathbf{w})}{\partial w_n} \\ &= \mathbf{w}' \frac{\partial f(\mathbf{w})}{\partial \mathbf{w}}, \end{aligned}$$

where

$$\underset{(n \times 1)}{\frac{\partial f(\mathbf{w})}{\partial \mathbf{w}}} = \begin{pmatrix} \frac{\partial f(\mathbf{w})}{\partial w_1} \\ \vdots \\ \frac{\partial f(\mathbf{w})}{\partial w_n} \end{pmatrix}$$

Verifying Euler's theorem

The function $f(w_1, w_2) = w_1 + w_2 = f(w) = w'\mathbf{1}$ is homogenous of degree one, and

$$\begin{aligned} \frac{\partial f(\mathbf{w})}{\partial w_1} &= \frac{\partial f(\mathbf{w})}{\partial w_2} = 1 \\ \frac{\partial f(\mathbf{w})}{\partial \mathbf{w}} &= \begin{pmatrix} \frac{\partial f(\mathbf{w})}{\partial w_1} \\ \frac{\partial f(\mathbf{w})}{\partial w_2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathbf{1} \end{aligned}$$

By Euler's theorem,

$$f(w) = w_1 \cdot 1 + w_2 \cdot 1 = w_1 + w_2 = \mathbf{w}'\mathbf{1}$$

The function $f(w_1, w_2) = (w_1^2 + w_2^2)^{1/2} = f(w) = (w'w)^{1/2}$ is homogenous of degree one, and

$$\begin{aligned}
\frac{\partial f(\mathbf{w})}{\partial w_1} &= \frac{1}{2}\left(w_1^2 + w_2^2\right)^{-1/2} 2w_1 = w_1 \left(w_1^2 + w_2^2\right)^{-1/2}, \\
\frac{\partial f(\mathbf{w})}{\partial w_2} &= \frac{1}{2}\left(w_1^2 + w_2^2\right)^{-1/2} 2w_2 = w_2 \left(w_1^2 + w_2^2\right)^{-1/2}.
\end{aligned}$$

By Euler's theorem

$$\begin{aligned}
f(\mathbf{w}) &= w_1 \cdot w_1 \left(w_1^2 + w_2^2\right)^{-1/2} + w_2 \cdot w_2 \left(w_1^2 + w_2^2\right)^{-1/2} \\
&= \left(w_1^2 + w_2^2\right)\left(w_1^2 + w_2^2\right)^{-1/2} \\
&= \left(w_1^2 + w_2^2\right)^{1/2}.
\end{aligned}$$

Using matrix algebra we have

$$\begin{aligned}
\frac{\partial f(\mathbf{w})}{\partial \mathbf{w}} &= \frac{\partial(\mathbf{w}'\mathbf{w})^{1/2}}{\partial \mathbf{w}} = \frac{1}{2}(\mathbf{w}'\mathbf{w})^{-1/2}\frac{\partial \mathbf{w}'\mathbf{w}}{\partial \mathbf{w}} \\
&= \frac{1}{2}(\mathbf{w}'\mathbf{w})^{-1/2} 2\mathbf{w} = (\mathbf{w}'\mathbf{w})^{-1/2} \cdot \mathbf{w}
\end{aligned}$$

so by Euler's theorem

$$\begin{aligned}
f(\mathbf{w}) &= \mathbf{w}'\frac{\partial f(\mathbf{w})}{\partial \mathbf{w}} = \mathbf{w}'(\mathbf{w}'\mathbf{w})^{-1/2} \cdot \mathbf{w} \\
&= (\mathbf{w}'\mathbf{w})^{-1/2}\mathbf{w}'\mathbf{w} = (\mathbf{w}'\mathbf{w})^{1/2}
\end{aligned}$$

General Risk Budgeting Result

Result: Because $RM(\mathbf{w})$ is a linearly homogenous function of $\mathbf{w}$, by Euler's Theorem

$$\begin{aligned} RM(\mathbf{w}) &= \sum_{i=1}^{n} w_i \frac{\partial RM(\mathbf{w})}{\partial w_i} \\ &= w_1 \frac{\partial RM(\mathbf{w})}{\partial w_1} + \cdots + w_n \frac{\partial RM(\mathbf{w})}{\partial w_n} \end{aligned}$$

Terminology

Asset i *marginal contribution to risk*

$$\frac{\partial RM(\mathbf{w})}{\partial w_i}$$

Asset i *contribution to risk*

$$w_i \frac{\partial RM(\mathbf{w})}{\partial w_i}$$

Asset i *percent contribution to risk*

$$\frac{w_i \frac{\partial RM(\mathbf{w})}{\partial w_i}}{RM(\mathbf{w})}$$

Analytic Results for $RM(\mathbf{w}) = \sigma(\mathbf{w})$

$$\begin{aligned}
R_p &= \mathbf{w}'\mathbf{R},\ var(\mathbf{R}) = \mathbf{\Sigma} \\
\sigma(\mathbf{w}) &= \left(\mathbf{w}'\mathbf{\Sigma}\mathbf{w}\right)^{1/2} \\
\frac{\partial \sigma(\mathbf{w})}{\partial \mathbf{w}} &= \frac{1}{\sigma(\mathbf{w})}\mathbf{\Sigma}\mathbf{w}
\end{aligned}$$

Note

$$\begin{aligned}
\mathbf{\Sigma}\mathbf{w} &= \begin{pmatrix} cov(R_1, R_p) \\ \vdots \\ cov(R_n, R_p) \end{pmatrix} = \sigma(\mathbf{w}) \begin{pmatrix} \beta_{1,p} \\ \vdots \\ \beta_{n,p} \end{pmatrix} \\
\beta_{i,p} &= cov(R_i, R_p)/\sigma^2(\mathbf{w})
\end{aligned}$$

Results for $RM(\mathbf{w}) = VaR_\alpha(\mathbf{w})$, $ES_\alpha(\mathbf{w})$

Gourieroux (2000) et al. and Scalliet (2002) showed that

$$\begin{aligned}
\frac{\partial VaR_\alpha(\mathbf{w})}{\partial w_i} &= E[R_i | R_p = VaR_\alpha(\mathbf{w})],\ i = 1, \ldots, n \\
\frac{\partial ES_\alpha(\mathbf{w})}{\partial w_i} &= E[R_i | R_p \leq VaR_\alpha(\mathbf{w})],\ i = 1, \ldots, n
\end{aligned}$$

Remarks

- Intuitive interpretation as stress loss scenario

- Analytic results are available under normality

Intiution

The portfolio return is

$$R_p = w'R = \sum_{i=1}^{n} w_i R_i$$

Then

$$\begin{aligned} VaR_\alpha(\mathbf{w}) &= E[R_p|R_p = VaR_\alpha] = \sum_{i=1}^{n} w_i E[R_i|R_p = VaR_\alpha] \\ ES_\alpha(\mathbf{w}) &= E[R_p|R_p \leq VaR_\alpha] = \sum_{i=1}^{n} w_i E[R_i|R_p \leq VaR_\alpha] \end{aligned}$$

Differentiating $VaR_\alpha(\mathbf{w})$ and $ES_\alpha(\mathbf{w})$ w.r.t. w_i then gives

$$\begin{aligned} \frac{\partial VaR_\alpha(\mathbf{w})}{\partial w_i} &= E[R_i|R_p = VaR_\alpha] \\ \frac{\partial ES_\alpha(\mathbf{w})}{\partial w_i} &= E[R_i|R_p \leq VaR_\alpha] \end{aligned}$$

Reverse Optimization, Implied Returns and Tail Risk Budgeting

- Standard portfolio optimization begins with a set of expected returns and risk forecasts.

- These inputs are fed into an optimization routine, which then produces the portfolio weights that maximizes some risk-to-reward ratio (typically subject to some constraints).

- Reverse optimization, by contrast, begins with a set of portfolio weights and risk forecasts, and then infers what the implied expected returns must be to satisfy optimality.

Optimized Portfolios

Suppose that the objective is to form a portfolio by maximizing a generalized expected return-to-risk (Sharpe) ratio:

$$\max_{\mathbf{w}} \frac{\mu_p(\mathbf{w})}{RM(\mathbf{w})}$$

$$\begin{aligned} \mu_p(\mathbf{w}) &= \mathbf{w}'\boldsymbol{\mu} \\ RM(\mathbf{w}) &= \text{linearly homogenous risk measure} \end{aligned}$$

The F.O.C.'s of the optimization are $(i = 1, \ldots, n)$

$$0 = \frac{\partial}{\partial w_i}\left(\frac{\mu_p(\mathbf{w})}{RM(\mathbf{w})}\right) = \frac{1}{RM(\mathbf{w})}\frac{\partial \mu_p(\mathbf{w})}{\partial w_i} - \frac{\mu_p(\mathbf{w})}{RM(\mathbf{w})^2}\frac{\partial RM(\mathbf{w})}{\partial w_i}$$

Reverse Optimization and Implied Returns

Reverse optimization uses the above optimality condition with fixed portfolio weights to determine the optimal fund expected returns. These optimal expected returns are called *implied returns*. The implied returns satisfy

$$\mu_i^{\text{implied}}(\mathbf{w}) = \frac{\mu_p(\mathbf{w})}{RM(\mathbf{w})} \times \frac{\partial RM(\mathbf{w})}{\partial w_i}$$

Result: fund i's implied return is proportional to its marginal contribution to risk, with the constant of proportionality being the generalized Sharpe ratio of the portfolio.

How to Use Implied Returns

- For a given generalized portfolio Sharpe ratio, $\mu_i^{\text{implied}}(\mathbf{w})$ is large if $\frac{\partial RM(\mathbf{w})}{\partial w_i}$ is large.
- If the actual or forecast expected return for fund i is less than its implied return, then one should reduce one's holdings of that asset
- If the actual or forecast expected return for fund i is greater than its implied return, then one should increase one's holdings of that asset