AMATH 546/ECON 589
Risk Budgeting

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Outline

- Portfolio Calculations
- Risk Budgeting
- Reverse Optimization and Implied Returns
Portfolio Risk Budgeting

- Additively decompose (slice and dice) portfolio risk measures into asset contributions

- Allow portfolio manager to know sources of asset risk for allocation and hedging purposes

- Allow risk manager to evaluate portfolio from asset risk perspective
Portfolio Calculations

Let $R_1, \ldots, R_n$ denote simple returns on $n$ assets, and let $w_1, \ldots, w_n$ denote portfolio weights such that $\sum_{i=1}^{n} w_i = 1$.

Portfolio return:

$$
\mathbf{R} = (R_1, \ldots, R_N), \ \mathbf{w} = (w_1, \ldots, w_n)', \ 1 = (1, \ldots, 1)'
$$

$$
R_p = \mathbf{w}'\mathbf{R} = \sum_{i=1}^{N} w_i R_i, \ \mathbf{w}'1 = 1
$$
Portfolio mean and variance:

Let \( \mathbf{R} \) be a random vector with

\[
E[\mathbf{R}] = \mu = (\mu_1, \ldots, \mu_n)'
\]

\[
\text{var}(\mathbf{R}) = E[(\mathbf{R} - \mu)(\mathbf{R} - \mu)'] = \Sigma = \begin{pmatrix}
\sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\
\sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{1n} & \sigma_{2n} & \cdots & \sigma_n^2
\end{pmatrix}
\]

Then

\[
\mu_p = \mathbf{w}'\mu, \quad \sigma_p^2 = \mathbf{w}'\Sigma\mathbf{w} \quad \text{and} \quad \sigma_p = \left(\mathbf{w}'\Sigma\mathbf{w}\right)^{1/2}
\]
Example: Portfolio risk decomposition for 2 risky asset portfolio

\[ R_p = w_1 R_1 + w_2 R_2 \]
\[ \sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1 w_2 \sigma_{12} \]
\[ \sigma_p = \left( w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1 w_2 \sigma_{12} \right)^{1/2} \]

Goal: Additively decompose \( \sigma_p^2 \) and \( \sigma_p \) into contributions from asset 1 and asset 2

\[ \sigma_p^2 = CR_1 + CR_2 \]
\[ \sigma_p = CR_1 + CR_2 \]
To get an additive decomposition for $\sigma_p^2$ write

$$
\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1 w_2 \sigma_{12} \\
= (w_1^2 \sigma_1^2 + w_1 w_2 \sigma_{12}) + (w_2^2 \sigma_2^2 + w_1 w_2 \sigma_{12})
$$

Here we can split the covariance contribution $2w_1 w_2 \sigma_{12}$ to portfolio variance evenly between the two assets and define

$$
w_1^2 \sigma_1^2 + w_1 w_2 \sigma_{12} = \text{variance contribution of asset 1} \\
w_2^2 \sigma_2^2 + w_1 w_2 \sigma_{12} = \text{variance contribution of asset 2}
$$
We can also define an additive decomposition for \( \sigma_p \)

\[
\sigma_p = \frac{w_1^2 \sigma_1^2 + w_1 w_2 \sigma_{12}}{\sigma_p} + \frac{w_2^2 \sigma_2^2 + w_1 w_2 \sigma_{12}}{\sigma_p}
\]

\[
\frac{w_1^2 \sigma_1^2 + w_1 w_2 \sigma_{12}}{\sigma_p} = \text{sd contribution of asset 1}
\]

\[
\frac{w_2^2 \sigma_2^2 + w_1 w_2 \sigma_{12}}{\sigma_p} = \text{sd contribution of asset 2}
\]
Euler’s Theorem and Risk Decompositions

- When we used $\sigma_p$ to measure portfolio risk, we were able to easily derive an additive risk decomposition.

- If we measure portfolio risk by VaR or ES it is not so obvious how to define individual asset risk contributions.

- For portfolio risk measures that are homogenous functions of degree one in the portfolio weights, Euler’s theorem provides a general method for additively decomposing risk into asset specific contributions.
Homogenous functions and Euler’s theorem

First we define a homogenous function of degree one.

**Definition 1** homogenous function of degree one

Let \( f(w_1, \ldots, w_n) \) be a continuous and differentiable function of the variables \( w_1, \ldots, w_n \). \( f \) is homogeneous of degree one if for any constant \( c > 0 \), \( f(c \cdot w_1, \ldots, c \cdot w_n) = c \cdot f(w_1, \ldots, w_n) \).

Note: In matrix notation we have \( f(w_1, \ldots, w_n) = f(w) \) where \( w = (w_1, \ldots, w_n)' \). Then \( f \) is homogeneous of degree one if \( f(c \cdot w) = c \cdot f(w) \)
Examples

Let $f(w_1, w_2) = w_1 + w_2$. Then

$$f(c \cdot w_1, c \cdot w_2) = c \cdot w_1 + c \cdot w_2 = c \cdot (w_1 + w_2) = c \cdot f(w_1, w_2)$$

Let $f(w_1, w_2) = w_1^2 + w_2^2$. Then

$$f(c \cdot w_1, c \cdot w_2) = c^2 w_1^2 + c^2 w_2^2 = c^2(w_1^2 + w_2^2) \neq c \cdot f(w_1, w_2)$$

Let $f(w_1, w_2) = \sqrt{w_1^2 + w_2^2}$. Then

$$f(c \cdot w_1, c \cdot w_2) = \sqrt{c^2 w_1^2 + c^2 w_2^2} = c \sqrt{w_1^2 + w_2^2} = c \cdot f(w_1, w_2)$$
Repeat examples using matrix notation

Define \( w = (w_1, w_2)' \) and \( 1 = (1, 1)' \).

Let \( f(w_1, w_2) = w_1 + w_2 = w'1 = f(w) \). Then

\[
f(c \cdot w) = (c \cdot w)'1 = c \cdot (w'1) = c \cdot f(w).
\]

Let \( f(w_1, w_2) = w_1^2 + w_2^2 = w'w = f(w) \). Then

\[
f(c \cdot w) = (c \cdot w)'(c \cdot w) = c^2 \cdot w'w \neq c \cdot f(w).
\]

Let \( f(w_1, w_2) = \sqrt{w_1^2 + w_2^2} = (w'w)^{1/2} = f(w) \). Then

\[
f(c \cdot w) = \left( (c \cdot w)'(c \cdot w) \right)^{1/2} = c \cdot \left( w'w \right)^{1/2} = c \cdot f(w).
\]
Consider a portfolio of \( n \) assets \( w = (w_1, \ldots, w_n)' \) with initial value \( V_0 \) and let \( \alpha \in (0, 1) \) denote a confidence level

\[
\begin{align*}
\mathbf{R} &= (R_1, \ldots, R_n)', \ w = (w_1, \ldots, w_n)' \\
E[\mathbf{R}] &= \mu, \ \text{cov}(\mathbf{R}) = \Sigma, \ \mathbf{R} \sim N(\mu, \Sigma)
\end{align*}
\]

Define

\[
\begin{align*}
R_p &= R_p(w) = w'\mathbf{R}, \\
\mu_p &= \mu_p(w) = w'\mu, \ \sigma_p^2 = \sigma_p^2(w) = w'\Sigma w, \\
\sigma_p &= \sigma_p(w) = (w'\Sigma w)^{1/2} \\
q_{1-\alpha}^{R_p} &= q_{1-\alpha}^{R_p}(w) = \mu_p(w) + \sigma_p(w) \times q_{1-\alpha}^Z \\
\text{VaR}_{\alpha}(w) &= -q_{1-\alpha}^{R_p}(w) \times V_0 \\
ES_{\alpha}(w) &= -V_0 \left( \mu_p(w) + \sigma_p(w) \times \frac{\phi(q_{1-\alpha}^Z)}{1 - \alpha} \right)
\end{align*}
\]
**Result:** Portfolio return $R_p(w)$, expected return $\mu_p(w)$, standard deviation $\sigma_p(w)$, normal quantile $q_{1-\alpha}^{R_p}(w)$, and normal VaR $\text{VaR}_\alpha(w)$, and normal ES $\text{ES}_\alpha(w)$ are homogenous functions of degree one in the portfolio weight vector $w$.

**Remarks**

- Above results for VaR and ES are based on assuming normally distributed returns

- It can be shown that linear homogeneity of VaR and ES holds for any distribution of returns
Let $RM(w)$ denote the risk measures $\sigma$, $VaR_\alpha$ and $ES_\alpha$ defined from returns as functions of the portfolio weights $w$.

**Result**: $RM(w)$ is a linearly homogenous function of $w$ for $RM = \sigma$, $VaR_\alpha$ and $ES_\alpha$. That is, $RM(c \cdot w) = c \cdot RM(w)$ for any constant $c \geq 0$. 


Theorem 2 Euler’s theorem

Let $f(w_1, \ldots, w_n) = f(w)$ be a continuous, differentiable and homogenous of degree one function of the variables $w = (w_1, \ldots, w_n)$. Then

$$f(w) = w_1 \cdot \frac{\partial f(w)}{\partial w_1} + w_2 \cdot \frac{\partial f(w)}{\partial w_2} + \cdots + w_n \cdot \frac{\partial f(w)}{\partial w_n}$$

$$= w' \frac{\partial f(w)}{\partial w},$$

where

$$\frac{\partial f(w)}{\partial w} = \begin{pmatrix} \frac{\partial f(w)}{\partial w_1} \\ \vdots \\ \frac{\partial f(w)}{\partial w_n} \end{pmatrix}$$
Verifying Euler’s theorem

The function \( f(w_1, w_2) = w_1 + w_2 = f(w) = w'1 \) is homogenous of degree one, and

\[
\frac{\partial f(w)}{\partial w_1} = \frac{\partial f(w)}{\partial w_2} = 1
\]

\[
\frac{\partial f(w)}{\partial w} = \begin{pmatrix} \frac{\partial f(w)}{\partial w_1} \\ \frac{\partial f(w)}{\partial w_2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1
\]

By Euler’s theorem,

\[
f(w) = w_1 \cdot 1 + w_2 \cdot 1 = w_1 + w_2 = w'1
\]
The function $f(w_1, w_2) = (w_1^2 + w_2^2)^{1/2} = f(w) = (w'w)^{1/2}$ is homogenous of degree one, and
\[
\frac{\partial f(w)}{\partial w_1} = \frac{1}{2} (w_1^2 + w_2^2)^{-1/2} 2w_1 = w_1 (w_1^2 + w_2^2)^{-1/2}, \\
\frac{\partial f(w)}{\partial w_2} = \frac{1}{2} (w_1^2 + w_2^2)^{-1/2} 2w_2 = w_2 (w_1^2 + w_2^2)^{-1/2}.
\]
By Euler’s theorem
\[
f(w) = w_1 \cdot w_1 (w_1^2 + w_2^2)^{-1/2} + w_2 \cdot w_2 (w_1^2 + w_2^2)^{-1/2} \\
= (w_1^2 + w_2^2) (w_1^2 + w_2^2)^{-1/2} \\
= (w_1^2 + w_2^2)^{1/2}.
\]
Using matrix algebra we have

\[
\frac{\partial f(w)}{\partial w} = \frac{\partial (w'w)^{1/2}}{\partial w} = \frac{1}{2} (w'w)^{-1/2} \frac{\partial w'w}{\partial w}
\]

\[
= \frac{1}{2} (w'w)^{-1/2} 2w = (w'w)^{-1/2} \cdot w
\]

so by Euler’s theorem

\[
f(w) = w'\frac{\partial f(w)}{\partial w} = w'(w'w)^{-1/2} \cdot w
\]

\[
= (w'w)^{-1/2} w'w = (w'w)^{1/2}
\]
General Risk Budgeting Result

Result: Because $RM(w)$ is a linearly homogenous function of $w$, by Euler’s Theorem

$$RM(w) = \sum_{i=1}^{n} w_i \frac{\partial RM(w)}{\partial w_i}$$

$$= w_1 \frac{\partial RM(w)}{\partial w_1} + \cdots + w_n \frac{\partial RM(w)}{\partial w_n}$$
**Terminology**

Asset $i$ marginal contribution to risk

$$\frac{\partial RM(w)}{\partial w_i}$$

Asset $i$ contribution to risk

$$w_i \frac{\partial RM(w)}{\partial w_i}$$

Asset $i$ percent contribution to risk

$$\frac{w_i \frac{\partial RM(w)}{\partial w_i}}{RM(w)}$$
Analytic Results for $RM(w) = \sigma(w)$

\[
R_p = w'R, \quad \text{var}(R) = \Sigma \\
\sigma(w) = (w'\Sigma w)^{1/2} \\
\frac{\partial \sigma(w)}{\partial w} = \frac{1}{\sigma(w)}\Sigma w
\]

Note

\[
\Sigma w = \begin{pmatrix} \text{cov}(R_1, R_p) \\ \vdots \\ \text{cov}(R_n, R_p) \end{pmatrix} = \sigma(w) \begin{pmatrix} \beta_{1,p} \\ \vdots \\ \beta_{n,p} \end{pmatrix} \\
\beta_{i,p} = \text{cov}(R_i, R_p)/\sigma^2(w)
\]
Results for $RM(w) = VaR_\alpha(w), \ ES_\alpha(w)$

Gourieroux (2000) et al. and Scalliet (2002) showed that

$$\frac{\partial VaR_\alpha(w)}{\partial w_i} = E[R_i|R_p = VaR_\alpha(w)], \ i = 1, \ldots, n$$

$$\frac{\partial ES_\alpha(w)}{\partial w_i} = E[R_i|R_p \leq VaR_\alpha(w)], \ i = 1, \ldots, n$$

Remarks

- Intuitive interpretation as stress loss scenario

- Analytic results are available under normality
Intuition

The portfolio return is

$$R_p = w'\mathbf{R} = \sum_{i=1}^{n} w_i R_i$$

Then

$$VaR_\alpha(w) = E[R_p|R_p = VaR_\alpha] = \sum_{i=1}^{n} w_i E[R_i|R_p = VaR_\alpha]$$

$$ES_\alpha(w) = E[R_p|R_p \leq VaR_\alpha] = \sum_{i=1}^{n} w_i E[R_i|R_p \leq VaR_\alpha]$$
Differentiating $VaR_\alpha(w)$ and $ES_\alpha(w)$ w.r.t. $w_i$ then gives

\[
\frac{\partial VaR_\alpha(w)}{\partial w_i} = E[R_i | R_p = VaR_\alpha]
\]

\[
\frac{\partial ES_\alpha(w)}{\partial w_i} = E[R_i | R_p \leq VaR_\alpha]
\]
Reverse Optimization, Implied Returns and Tail Risk Budgeting

• Standard portfolio optimization begins with a set of expected returns and risk forecasts.

• These inputs are fed into an optimization routine, which then produces the portfolio weights that maximizes some risk-to-reward ratio (typically subject to some constraints).

• Reverse optimization, by contrast, begins with a set of portfolio weights and risk forecasts, and then infers what the implied expected returns must be to satisfy optimality.
Optimized Portfolios

Suppose that the objective is to form a portfolio by maximizing a generalized expected return-to-risk (Sharpe) ratio:

$$\max_w \frac{\mu_p(w)}{RM(w)}$$

$$\mu_p(w) = w'\mu$$

$$RM(w) = \text{linearly homogenous risk measure}$$

The F.O.C.'s of the optimization are \((i = 1, \ldots, n)\)

$$0 = \frac{\partial}{\partial w_i} \left( \frac{\mu_p(w)}{RM(w)} \right) = \frac{1}{RM(w)} \frac{\partial \mu_p(w)}{\partial w_i} - \frac{\mu_p(w)}{RM(w)^2} \frac{\partial RM(w)}{\partial w_i}$$
Reverse Optimization and Implied Returns

Reverse optimization uses the above optimality condition with fixed portfolio weights to determine the optimal fund expected returns. These optimal expected returns are called *implied returns*. The implied returns satisfy

\[ \mu_i^{\text{implied}}(\mathbf{w}) = \frac{\mu_p(\mathbf{w})}{RM(\mathbf{w})} \times \frac{\partial RM(\mathbf{w})}{\partial w_i} \]

**Result:** fund i’s implied return is proportional to its marginal contribution to risk, with the constant of proportionality being the generalized Sharpe ratio of the portfolio.
How to Use Implied Returns

- For a given generalized portfolio Sharpe ratio, $\mu_i^{\text{implied}}(w)$ is large if $\frac{\partial RM(w)}{\partial w_i}$ is large.

- If the actual or forecast expected return for fund $i$ is less than its implied return (based on a convex risk measure), then one should reduce one’s holdings of that asset.

- If the actual or forecast expected return for fund $i$ is greater than its implied return, then one should increase one’s holdings of that asset.