Financial Econometrics and Quantitative Risk Managenent
Return Properties

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Lecture Outline

- Course introduction
- Return definitions
- Empirical properties of returns

Reading

- FRF chapter 1
- FMUND chapter 1 and chapter 2
- SDAFE chapter 2 and chapter 4

Discrete Returns

Simple Net Return

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} = \%\Delta P_t$$

Gross Return

$$1 + R_t = \frac{P_t}{P_{t-1}}$$

2−Period Return

$$R_t(2) = \frac{P_t - P_{t-2}}{P_{t-2}} = \frac{P_t}{P_{t-2}} - 1$$

$$\begin{aligned} R_t(2) &= \frac{P_t}{P_{t-1}} \cdot \frac{P_{t-1}}{P_{t-2}} - 1 \\ &= (1 + R_t)(1 + R_{t-1}) - 1. \end{aligned}$$

$k-$ Period Return

$$\begin{aligned} 1 + R_t(k) &= (1 + R_t)(1 + R_{t-1}) \cdots (1 + R_{t-k+1}) \\ &= \prod_{j=0}^{k-1} (1 + R_{t-j}). \end{aligned}$$

Adjusting for Dividends (Total Returns)

$$R_t = \frac{P_t + D_t - P_{t-1}}{P_{t-1}} = \frac{P_t - P_{t-1}}{P_{t-1}} + \frac{D_t}{P_{t-1}}$$

Adjusting for Inflation (Real Returns)

$$1 + R_t^{\text{Real}} = \frac{P_t}{P_{t-1}} \cdot \frac{CPI_{t-1}}{CPI_t}$$

Portfolio Return

$$
\begin{aligned}
P_{p,t} &= \sum_{i=1}^{n} w_i P_{i,t}, \sum_{i=1}^{n} w_i = 1 \\
R_{p,t} &= \sum_{i=1}^{n} w_i R_{i,t}
\end{aligned}
$$

Excess Returns

$$
\begin{aligned}
Z_t &= R_t - R_{ft} \\
R_{ft} &= \text{T-bill rate or LIBOR rate}
\end{aligned}
$$

Continuously Compounded Returns

$$
\begin{aligned}
r_t &= \ln(1+R_t) = \ln\left(\frac{P_t}{P_{t-1}}\right) \\
&= \ln(P_t) - \ln(P_{t-1}) \\
&= p_t - p_{t-1} \\
e^{r_t} &= 1 + R_t = \frac{P_t}{P_{t-1}} \\
&\Longrightarrow P_t = P_{t-1}e^{r_t}
\end{aligned}
$$

Note:

$$R_t = e^{r_t} - 1$$

2−period return

$$
\begin{aligned}
r_t(2) &= \ln(1 + R_t(2)) = \ln\left(\frac{P_t}{P_{t-2}}\right) = p_t - p_{t-2} \\
&= \ln\left(\frac{P_t}{P_{t-1}} \cdot \frac{P_{t-1}}{P_{t-2}}\right) \\
&= \ln\left(\frac{P_t}{P_{t-1}}\right) + \ln\left(\frac{P_{t-1}}{P_{t-2}}\right) \\
&= r_t + r_{t-1}.
\end{aligned}
$$

$k-$period return

$$
\begin{aligned}
r_t(k) &= \ln(1 + R_t(k)) = \ln\left(\frac{P_t}{P_{t-k}}\right) = p_t - p_{t-k} \\
&= \sum_{j=0}^{k-1} r_{t-j}
\end{aligned}
$$

Adjusting for Dividends (Total Returns)

$$
\begin{aligned}
r_t &= \ln(1 + R_t) = \ln\left(\frac{P_t + D_t}{P_{t-1}}\right) \\
&= \ln(P_t + D_t) - \ln(P_{t-1})
\end{aligned}
$$

Adjusting for Inflation (Real Returns)

$$
\begin{aligned}
r_t^{\text{Real}} &= \ln(1 + R_t^{\text{Real}}) = \ln\left(\frac{P_t}{P_{t-1}} \cdot \frac{CPI_{t-1}}{CPI_t}\right) \\
&= r_t - \pi_t
\end{aligned}
$$

Portfolio Return

$$\begin{aligned} r_{t,p} &= \ln(1 + R_{t,p}) \\ &= \ln\left(1 + \sum_{i=1}^{n} w_i R_{i,t}\right) \\ &\neq \sum_{i=1}^{n} w_i r_{i,t} \end{aligned}$$

But

$$r_{t,p} \approx \sum_{i=1}^{n} w_i r_{i,t} \text{ if } R_{i,t} \text{ is small}$$

Excess Returns

$$
\begin{aligned}
Z_t &= R_t - R_{ft} \\
z_t &= \ln(Z_t) = \ln(R_t - R_{ft}) \neq r_t - r_{ft}
\end{aligned}
$$

But if Z_t is small then

$$z_t \approx r_t - r_{ft}$$

Stylized Facts of Asset Return Distributions

- *Fat tails*
- *Asymmetry*
- *Aggregated normality*
- *Absence of serial correlation*
- *Volatility clustering*
- *Time-varying cross correlation*

Shape Characteristics

Let $\tilde{r}$ be a random variable with pdf f

$$\begin{aligned} \mu &= E[r] : \text{ center} \\ \sigma^2 &= \text{var}(r) = E[(r-\mu)^2] : \text{ spread} \\ \text{skew}(r) &= E\left[\frac{(r-\mu)^3}{\sigma^3}\right] : \text{ symmetry} \\ \text{kurt}(r) &= E\left[\frac{(r-\mu)^4}{\sigma^4}\right] : \text{ tail thickness} \end{aligned}$$

Note: The k^{th} moment and central moment of $\tilde{r}$ is

$$\begin{aligned} m_k' &= E[\tilde{r}^k] \\ m_k &= E[(\tilde{r}-\mu)^k] \end{aligned}$$

Normal Distribution

$X \sim N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad -\infty \leq x \leq \infty$$

$$\begin{aligned} E[X] &= \mu \\ \mathrm{var}(X) &= \sigma^2 \\ \mathrm{skew}(X) &= 0 \\ \mathrm{kurt}(X) &= 3 \\ m_k &= 0 \text{ for } k \text{ odd} \end{aligned}$$

Sample moments

Let $\{r_t, \ldots, r_T\}$ denote a random sample of size T where r_t is a realization of the random variable $\tilde{r}$.

$$
\begin{aligned}
\hat{\mu} &= \frac{1}{T}\sum_{t=1}^{T} r_t,\ \hat{\sigma}^2 = \frac{1}{T-1}\sum_{t=1}^{T}(r_t - \hat{\mu})^2 = \hat{m}_2 \\
\widehat{\mathsf{skew}} &= \frac{\hat{m}_3}{\hat{\sigma}^3},\ \widehat{\mathsf{kurt}} = \frac{\hat{m}_4}{\hat{\sigma}^3} \\
\hat{m}_k &= \frac{1}{T-1}\sum_{t=1}^{T}(r_t - \hat{\mu})^k,
\end{aligned}
$$

Note: we divide by $T-1$ to get unbiased estimates. Check software to see how moments are computed.

Testing for Normality

- QQ-plot: plot standardized empirical quantiles vs. theoretical quantiles from specified distribution. Note: Shapiro-Wilks (SW) test for normality: correlation coefficient between values used in QQ-plot

- Jarque-Bera (JB) test for normality

$$\text{JB} = \frac{T}{6}\left(\widehat{\text{skew}}^2 + \frac{(\widehat{\text{kurt}} - 3)^2}{4}\right) \overset{A}{\sim} \chi^2(2)$$

Note: if $\tilde{r} \sim N(\mu, \sigma^2)$ then

$$\sqrt{T}\widehat{\text{skew}} \overset{A}{\sim} N(0, 6), \ \sqrt{T}(\widehat{\text{kurt}} - 3) \overset{A}{\sim} N(0, 24)$$

- Kolmogorov-Smirnov (KS) test compares the empirical CDF of returns with the CDF of the normal distribution (or any other assumed distribution)
  - Sort returns: $r_{(1)} \leq \cdots \leq r_{(T)}$ and compute empirical CDF $\hat{F}_r(r_{(t)}) = t/T$
  - Evaluate normal CDF: $\Phi\left(\frac{r_{(t)}-\hat{\mu}}{\hat{\sigma}}\right)$
  - Compute KS statistic: $KS = \sup_t \left|\Phi\left(\frac{r_{(t)}-\hat{\mu}}{\hat{\sigma}}\right) - t/T\right|$

Student's-t distribution

Let $Z \sim N(0,1)$, $W \sim \chi^2(v)$ such that Z and W are independent. Then

$$X = \frac{Z}{\sqrt{W/v}} \sim t_v$$

where t_v denotes a (standardized) Student's t distribution with v degrees of freedom. Note:

$$\begin{aligned} E[X] &= 0,\ \text{var}(X) = \frac{v}{v-2},\ v > 2 \\ \text{skew} &= 0,\ \text{kurt} - 3 = \frac{6}{\nu - 4},\ v > 4 \end{aligned}$$

Existence of moments depends on degrees of freedom (df) parameter ν. Cauchy = Student's-t with 1 df. Only density exists.

If $X \sim t_v$ then

$$Y = \mu + \frac{\sigma X}{\sqrt{v/(v-2)}}$$

has moments

$$E[Y] = \mu, \ \text{var}(Y) = \sigma^2$$

Density function

$$
\begin{aligned}
f(x;v) &= \left[\frac{\Gamma\{(v+1)/2}{(\pi v)^{1/2}\Gamma(v/2)}\right]\frac{1}{\{1+(x^2/v)\}^{(v+1)/2}} \\
\Gamma(t) &= \int_0^\infty x^{t-1}\exp(-x)dx = \text{ gamma function}
\end{aligned}
$$

The d.f. parameter v can be estimated by MLE.

Note: A simple method of moments estimator for v is based on kurtosis:

$$
\text{kurt} - 3 = \frac{6}{\nu - 4} \Rightarrow \nu = 6/(\text{kurt} - 3) + \ 4
$$

Skew Normal Distribution

Azzalini and Capitanio (2002) define $Z \sim SN(\xi, \omega, \alpha)$ as a skew-normal random variable with density

$$\begin{aligned} f_Z(z) &= 2\phi(z-\xi)\Phi(\alpha\omega^{-1}(z-\xi)) \\ \phi(z) &= (2\pi)^{-1/2}\exp\left(-\frac{z^2}{2}\right),\ \Phi(z) = \int_{-\infty}^{z}\phi(x)dx \\ \xi &= \text{ location parameter, } -\infty < \xi < \infty \\ \omega &= \text{ scale parameter, } \omega > 0 \\ \alpha &= \text{ shape (skew) parameter, } -\infty < \alpha < \infty \end{aligned}$$

Note: Estimation and simulation functions in R package sn

Remarks

- $\alpha = 0$, $Z \sim N(\xi, \omega^2)$
- $\alpha > 0 \Rightarrow$ positive skewness
- $\alpha < 0 \Rightarrow$ negative skewness

Skew-t Distribution

Azzalini and Capitanio (2002) define $Y \sim St(\xi, \omega, \alpha, v)$ as a skew-t random variable using the transformation

$$\begin{aligned} Y &= \xi + V^{-1/2} Z \\ & Z \sim SN(\xi, \omega, \alpha) \\ & V \sim \chi^2(v)/v \end{aligned}$$

The parameters ξ, ω and α have the same interpretation as in the skew-normal and

$$v = \text{ degrees of freedom parameter, } v > 0$$

Note: Estimation and simulation functions in R package sn

Defn: The stochastic process $\{\tilde{r}_t\}$ is *covariance stationary* if

$$\begin{aligned} E[\tilde{r}_t] &= \mu \text{ for all } t \\ cov(\tilde{r}_t, \tilde{r}_{t-j}) &= E[(\tilde{r}_t - \mu)(\tilde{r}_{t-j} - \mu)] = \gamma_j \text{ for all } t \text{ and any } j \end{aligned}$$

The parameter γ_j is called the j^{th} order or lag j *autocovariance* of $\{\tilde{r}_t\}$

The *autocorrelations* of $\{\tilde{r}_t\}$ are defined by

$$\rho_j = \frac{cov(\tilde{r}_t, \tilde{r}_{t-j})}{\sqrt{var(\tilde{r}_t) var(\tilde{r}_{t-j})}} = \frac{\gamma_j}{\gamma_0}$$

and a plot of ρ_j against j is called the *autocorrelation function* (ACF)

The lag j *sample autocovariance* and lag j *sample autocorrelation* are defined as

$$\begin{aligned}\hat{\gamma}_j &= \frac{1}{T}\sum_{t=j+1}^{T}(r_t - \bar{r})(r_{t-j} - \bar{r}) \\ \hat{\rho}_j &= \frac{\hat{\gamma}_j}{\hat{\gamma}_0}\end{aligned}$$

where $\bar{r} = \frac{1}{T}\sum_{t=1}^{T} r_t$ is the sample mean.

The sample ACF (SACF) is a plot of $\hat{\rho}_j$ against j.

Example: White noise (GWN) processes

Perhaps the most simple stationary time series is the *independent Gaussian white noise* process $\{\tilde{r}_t\} \sim iid\ N(0, \sigma^2) \equiv GWN(0, \sigma^2)$. This process has $\mu = \gamma_j = \rho_j = 0$ $(j \neq 0)$.

Two slightly more general processes are the independent *white noise* (IWN) process, $\{\tilde{r}_t\} \sim IWN(0, \sigma^2)$, and the *white noise* (WN) process, $\{\tilde{r}_t\} \sim WN(0, \sigma^2)$.

Both processes have mean zero and variance σ^2, but the IWN process has independent increments, whereas the WN process has uncorrelated increments.

The SACF is typically shown with 95% confidence limits about zero. These limits are based on the result that if $\{\tilde{r}_t\} \sim iid\,(0, \sigma^2)$ then

$$\hat{\rho}_j \overset{A}{\sim} N\left(0, \frac{1}{T}\right),\ j > 0.$$

The notation $\hat{\rho}_j \overset{A}{\sim} N\left(0, \frac{1}{T}\right)$ means that the distribution of $\hat{\rho}_j$ is approximated by normal distribution with mean 0 and variance $\frac{1}{T}$ and is based on the central limit theorem result $\sqrt{T}\hat{\rho}_j \overset{d}{\to} N\,(0, 1)$. The 95% limits about zero are then $\pm\frac{1.96}{\sqrt{T}}$.

Testing for White Noise

Consider testing the null hypothesis

$$H_0 : \{\tilde{r}_t\} \sim WN(0, \sigma^2)$$

Under the null, all of the autocorrelations ρ_j for $j > 0$ are zero. To test this null, Box and Pierce (1970) suggested the *Q-statistic*

$$Q(k) = T \sum_{j=1}^{k} \hat{\rho}_j^2$$

Under the null, $Q(k)$ is asymptotically distributed $\chi^2(k)$. In a finite sample, the Q-statistic may not be well approximated by the $\chi^2(k)$. Ljung and Box (1978) suggested the *modified Q-statistic*

$$MQ(k) = T(T+2) \sum_{j=1}^{k} \frac{\hat{\rho}_j^2}{T-j}$$