Amath 546/Econ 589
Factor Model Risk Analysis

Eric Zivot
University of Washington

June 3, 2013

Outline

- Factor Model Specification
- Factor Risk Budgeting
- Portfolio Risk Budgeting
- Factor Model Monte Carlo

Introduction

Factor models for asset returns (equity, fixed income, hedge funds, etc.) are used to

- Decompose risk and return into explanable and unexplainable components
- Generate estimates of abnormal return
- Describe the covariance structure of returns
- Predict returns in specified stress scenarios
- Provide a framework for portfolio risk analysis

Three Types of Asset Return Factor Models

1. Macroeconomic factor model

    (a) Factors are observable economic and financial time series

2. Fundamental factor model

    (a) Factors are created from observerable asset characteristics

3. Statistical factor model

    (a) Factors are unobservable and extracted from asset returns

Factor Model Specification

The three types of multifactor models for asset returns have the general form

$$\begin{aligned} R_{it} &= \alpha_i + \beta_{1i} f_{1t} + \beta_{2i} f_{2t} + \cdots + \beta_{Ki} f_{Kt} + \varepsilon_{it} \\ &= \alpha_i + \boldsymbol{\beta}_i' \mathbf{f}_t + \varepsilon_{it} \end{aligned} \tag{1}$$

- R_{it} is the simple return (real or in excess of the risk-free rate) on asset i $(i = 1, \ldots, N)$ in time period t $(t = 1, \ldots, T)$,
- f_{kt} is the k^{th} *common factor* $(k = 1, \ldots, K)$,
- β_{ki} is the *factor loading* or *factor beta* for asset i on the k^{th} factor,
- ε_{it} is the asset *specific factor*.

Assumptions

1. The factor realizations, $\mathbf{f}_t$, are stationary with unconditional moments

$$\begin{aligned} E[\mathbf{f}_t] &= \boldsymbol{\mu}_f \\ cov(\mathbf{f}_t) &= E[(\mathbf{f}_t - \boldsymbol{\mu}_f)(\mathbf{f}_t - \boldsymbol{\mu}_f)'] = \underset{K\times K}{\boldsymbol{\Omega}_f} \end{aligned}$$

2. Asset specific error terms, ε_{it}, are uncorrelated with each of the common factors, f_{kt},

$$cov(f_{kt}, \varepsilon_{it}) = 0, \text{ for all } k, \ i \text{ and } t.$$

3. Error terms ε_{it} are serially uncorrelated and contemporaneously uncorre-

lated across assets

$$
\begin{aligned}
cov(\varepsilon_{it}, \varepsilon_{js}) &= \sigma_i^2 \text{ for all } i = j \text{ and } t = s \\
&= 0, \text{ otherwise}
\end{aligned}
$$

Remarks:

- Statistical modeling of returns involves statistical modeling of factors and residuals

- Typical factor models have a small number of factors (e.g., $K < 10$)

- Multivariate modeling of factors is a relatively low dimensional problem
  - Copula models are feasible for factors
  - Multivariate GARCH (e.g. DCC) is feasible for factor covariances

- $cov(\varepsilon_{it}, \varepsilon_{js}) = 0$ $(i \neq j) \Rightarrow$ only need univariate statistical models for ε_{it}

Notation

Vectors with a subscript t represent the cross-section of all assets

$$\underset{(N\times 1)}{\mathbf{R}_t} = \begin{pmatrix} R_{1t} \\ \vdots \\ R_{Nt} \end{pmatrix}, \; t = 1, \ldots, T$$

Vectors with a subscript i represent the time series of a given asset

$$\underset{(T\times 1)}{\mathbf{R}_i} = \begin{pmatrix} R_{i1} \\ \vdots \\ R_{iT} \end{pmatrix}, \; i = 1, \ldots, N$$

Matrix of all assets over all time periods (columns = assets, rows = time period)

$$\underset{(T\times N)}{\mathbf{R}} = \begin{pmatrix} R_{11} & \cdots & R_{N1} \\ \vdots & \ddots & \vdots \\ R_{1T} & \cdots & R_{NT} \end{pmatrix}$$

Cross Section Regression

The multifactor model (1) may be rewritten as a *cross-sectional* regression model at time t by stacking the equations for each asset to give

$$\begin{aligned} \underset{(N\times 1)}{\mathbf{R}_t} &= \underset{(N\times 1)}{\boldsymbol{\alpha}} + \underset{(N\times K)}{\mathbf{B}}\underset{(K\times 1)}{\mathbf{f}_t} + \underset{(N\times 1)}{\boldsymbol{\varepsilon}_t}, \; t = 1, \ldots, T \qquad (2) \\ \underset{(N\times K)}{\mathbf{B}} &= \begin{bmatrix} \boldsymbol{\beta}_1' \\ \vdots \\ \boldsymbol{\beta}_N' \end{bmatrix} = \begin{bmatrix} \beta_{11} & \cdots & \beta_{1K} \\ \vdots & \ddots & \vdots \\ \beta_{N1} & \cdots & \beta_{NK} \end{bmatrix} \\ E[\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t' | \mathbf{f}_t] &= \mathbf{D} = diag(\sigma_1^2, \ldots, \sigma_N^2) \end{aligned}$$

Note: Cross-sectional heteroskedasticity

This representation is useful for risk analysis across assets.

Time Series Regression

The multifactor model (1) may also be rewritten as a *time-series* regression model for asset i by stacking observations for a given asset i to give

$$\underset{(T\times 1)}{\mathbf{R}_i} = \underset{(T\times 1)}{\mathbf{1}_T}\underset{(1\times 1)}{\alpha_i} + \underset{(T\times K)}{\mathbf{F}}\underset{(K\times 1)}{\boldsymbol{\beta}_i} + \underset{(T\times 1)}{\boldsymbol{\varepsilon}_i}, \; i = 1, \ldots, N \qquad (3)$$

$$\underset{(T\times K)}{\mathbf{F}} = \begin{bmatrix} \mathbf{f}_1' \\ \vdots \\ \mathbf{f}_T' \end{bmatrix} = \begin{bmatrix} f_{11} & \cdots & f_{K1} \\ \vdots & \ddots & \vdots \\ f_{1T} & \cdots & f_{KT} \end{bmatrix}$$

$$E[\boldsymbol{\varepsilon}_i \boldsymbol{\varepsilon}_i'] = \sigma_i^2 \mathbf{I}_T$$

Note: Time series homoskedasticity

This representation is useful for estimating α_i and β_i using linear regression

Multivariate Regression

Collecting data from $i = 1, \ldots, N$ allows the model (3) to be expressed as the multivariate regression

$$[\mathbf{R}_1, \ldots, \mathbf{R}_N] = \mathbf{1}_T[\alpha_1, \ldots, \alpha_N] + \mathbf{F}[\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_N] + [\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_N]$$

or

$$\begin{aligned}
\underset{(T\times N)}{\mathbf{R}} &= \underset{(T\times 1)}{\mathbf{1}_T}\underset{(1\times N)}{\boldsymbol{\alpha}'} + \underset{(T\times K)}{\mathbf{F}}\underset{(K\times N)}{\mathbf{B}'} + \underset{(T\times N)}{\mathbf{E}} \\
&= \mathbf{X}\boldsymbol{\Gamma}' + \mathbf{E} \\
\underset{(T\times(K+1))}{\mathbf{X}} &= [\mathbf{1}_T \vdots \mathbf{F}], \quad \underset{((K+1)\times N)}{\boldsymbol{\Gamma}'} = \begin{bmatrix} \boldsymbol{\alpha}' \\ \mathbf{B}' \end{bmatrix},
\end{aligned}$$

Alternatively, collecting data from $t = 1, \ldots, T$ allows the model (2) to be expressed as the multivariate regression

$$[\mathbf{R}_1, \ldots, \mathbf{R}_T] = [\boldsymbol{\alpha}, \ldots, \boldsymbol{\alpha}] + \mathbf{B}[\mathbf{f}_1, \ldots, \mathbf{f}_T] + [\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_T]$$

or

$$\begin{aligned}
\underset{(N\times T)}{\mathbf{R}'} &= \underset{(N\times 1)}{\boldsymbol{\alpha}}\underset{(1\times T)}{\mathbf{1}_T'} + \underset{(N\times K)}{\mathbf{B}}\underset{(K\times T)}{\mathbf{F}'} + \underset{(N\times T)}{\mathbf{E}'} \\
&= \boldsymbol{\Gamma}\mathbf{X}' + \mathbf{E}' \\
\underset{((K+1)\times T)}{\mathbf{X}'} &= \begin{bmatrix} \mathbf{1}_T' \\ \mathbf{F}' \end{bmatrix}, \underset{(N\times(K+1))}{\boldsymbol{\Gamma}} = [\boldsymbol{\alpha} \vdots \mathbf{B}],
\end{aligned}$$

Expected Return ($\alpha - \beta$) Decomposition

$$E[R_{it}] = \alpha_i + \boldsymbol{\beta}_i' E[\mathbf{f}_t]$$

- $\boldsymbol{\beta}_i' E[\mathbf{f}_t] =$ explained expected return due to systematic risk factors

- $\alpha_i = E[R_{it}] - \boldsymbol{\beta}_i' E[\mathbf{f}_t] =$ unexplained expected return (abnormal return)

Note: Equilibrium asset pricing models impose the restriction $\alpha_i = 0$ (no abnormal return) for all assets $i = 1, \ldots, N$

Covariance Structure

Using the cross-section regression

$$\underset{(N\times 1)}{\mathbf{R}_t} = \underset{(N\times 1)}{\boldsymbol{\alpha}} + \underset{(N\times K)}{\mathbf{B}}\underset{(K\times 1)}{\mathbf{f}_t} + \underset{(N\times 1)}{\boldsymbol{\varepsilon}_t},\ t = 1, \ldots, T$$

and the assumptions of the multifactor model, the $(N \times N)$ covariance matrix of asset returns has the form

$$cov(\mathbf{R}_t) = \boldsymbol{\Omega}_{FM} = \mathbf{B}\boldsymbol{\Omega}_f\mathbf{B}' + \mathbf{D} \tag{4}$$

Note, (4) implies that

$$\begin{aligned} var(R_{it}) &= \boldsymbol{\beta}_i'\boldsymbol{\Omega}_f\boldsymbol{\beta}_i + \sigma_i^2 \\ cov(R_{it}, R_{jt}) &= \boldsymbol{\beta}_i'\boldsymbol{\Omega}_f\boldsymbol{\beta}_j \\ corr(R_{it}, R_{jt}) &= \frac{\boldsymbol{\beta}_i'\boldsymbol{\Omega}_f\boldsymbol{\beta}_j}{\left[\left(\boldsymbol{\beta}_i'\boldsymbol{\Omega}_f\boldsymbol{\beta}_i + \sigma_i^2\right)(\boldsymbol{\beta}_j'\boldsymbol{\Omega}_f\boldsymbol{\beta}_j + \sigma_j^2)\right]^{1/2}} \end{aligned}$$

Conditional Covariance Structure

Let I_t denote the information available at time t. We can allow the factor covariances and residual variances to be time varying

$$\begin{aligned}\underset{k\times 1}{\mathbf{f}_t} &= \boldsymbol{\mu}_{t|t-1} + \boldsymbol{\varepsilon}_{f,t} \\ \boldsymbol{\varepsilon}_{f,t} &= \boldsymbol{\Omega}_{f,t}^{1/2}\mathbf{z}_{f,t} \Rightarrow var(\boldsymbol{\varepsilon}_{f,t}|I_{t-1}) = \underset{k\times k}{\boldsymbol{\Omega}_{f,t}} \\ \varepsilon_{it} &= \sigma_{i,t}z_{it} \Rightarrow var(\varepsilon_{it}|I_{t-1}) = \sigma_{i,t}^2,\ i = 1,\ldots,n\end{aligned}$$

Then the factor model conditional covariance matrix is

$$cov(\mathbf{R}_t|I_{t-1}) = \boldsymbol{\Omega}_{FM,t} = \mathbf{B}\boldsymbol{\Omega}_{f,t}\mathbf{B}' + \mathbf{D}_t$$

Note: We can also allow the factor betas to be time varying (i.e., $\mathbf{B} = \mathbf{B}_t$)

Portfolio Analysis

Let $\mathbf{w} = (w_1, \ldots, w_n)$ be a vector of portfolio weights (w_i = fraction of wealth in asset i). If $\mathbf{R}_t$ is the $(N \times 1)$ vector of simple returns then

$$R_{p,t} = \mathbf{w}'\mathbf{R}_t = \sum_{i=1}^{N} w_i R_{it}$$

Portfolio Factor Model

$$\begin{aligned}
\mathbf{R}_t &= \boldsymbol{\alpha} + \mathbf{B}\mathbf{f}_t + \boldsymbol{\varepsilon}_t \Rightarrow \\
R_{p,t} &= \mathbf{w}'\boldsymbol{\alpha} + \mathbf{w}'\mathbf{B}\mathbf{f}_t + \mathbf{w}'\boldsymbol{\varepsilon}_t = \alpha_p + \boldsymbol{\beta}_p'\mathbf{f}_t + \varepsilon_{p,t} \\
\alpha_p &= \mathbf{w}'\boldsymbol{\alpha},\ \boldsymbol{\beta}_p' = \mathbf{w}'\mathbf{B},\ \varepsilon_{p,t} = \mathbf{w}'\boldsymbol{\varepsilon}_t \\
var(R_{p,t}) &= \boldsymbol{\beta}_p'\boldsymbol{\Omega}_f\boldsymbol{\beta}_p + var(\varepsilon_{p,t}) = \mathbf{w}'\mathbf{B}\boldsymbol{\Omega}_f\mathbf{B}'\mathbf{w} + \mathbf{w}'\mathbf{D}\mathbf{w}
\end{aligned}$$

Active and Static Portfolios

- Active portfolios have weights that change over time due to active asset allocation decisions

- Static portfolios have weights that are fixed over time (e.g. equally weighted portfolio)

- Factor models can be used to analyze the risk of both active and static portfolios

Unconditional Asset Risk Measures: Factor Model and Normal Distribution

$$R_{it} = \alpha_i + \boldsymbol{\beta}_i' \mathbf{f}_t + \varepsilon_{it}$$
$$\mathbf{f}_t \sim iid\ N(\boldsymbol{\mu}_f, \boldsymbol{\Omega}_f),\ var(\varepsilon_{it}) = \sigma^2_{\varepsilon,i},\ cov(f_{k,t}, \varepsilon_{is}) = 0 \text{ for all } k, t, s$$

Then

$$\begin{aligned}
E[R_{it}] &= \mu_{FM,i} = \alpha_i + \boldsymbol{\beta}_i' \boldsymbol{\mu}_f \\
var(R_{it}) &= \sigma^2_{FM,i} = \boldsymbol{\beta}_i' \boldsymbol{\Omega}_f \boldsymbol{\beta}_i + \sigma^2_{\varepsilon,i} \\
\sigma_{FM,i} &= \sqrt{\boldsymbol{\beta}_i' \boldsymbol{\Omega}_f \boldsymbol{\beta}_i + \sigma^2_{\varepsilon,i}} \\
VaR_p^{N,FM} &= \mu_{FM,i} + \sigma_{FM,i} \times z_p \\
ETL_p^{N,FM} &= \mu_{FM,i} - \sigma_{FM,i} \frac{1}{p} \phi(z_p)
\end{aligned}$$

Note: In practice, $\alpha_i = 0$ is typically imposed so that $\mu_{FM,i} = \boldsymbol{\beta}_i' \boldsymbol{\mu}_f$.

Conditional Asset Risk Measures: Factor Model and Normal Distribution

$$\begin{aligned}
var(R_{it}|I_{t-1}) &= \sigma^2_{FM,i,t} = \boldsymbol{\beta}_i'\boldsymbol{\Omega}_{f,t}\boldsymbol{\beta}_i + \sigma^2_{\varepsilon,i,t} \\
\sigma_{FM,i,t} &= \sqrt{\boldsymbol{\beta}_i'\boldsymbol{\Omega}_{f,t}\boldsymbol{\beta}_i + \sigma^2_{\varepsilon,i,t}} \\
VaR^{N,FM}_{p,t} &= \mu_{FM,i,t} + \sigma_{FM,i,t} \times z_p \\
ETL^{N,FM}_{p,t} &= \mu_{FM,i,t} - \sigma_{FM,i,t}\frac{1}{p}\phi(z_p)
\end{aligned}$$

where $\boldsymbol{\Omega}_{f,t}$ is modeled as an EWMA or DCC and $\sigma^2_{\varepsilon,i,t}$ is modeled as an EWMA or GARCH.

Note 1: For daily data it is typically assumed that $\mu_{FM,i,t} = 0$.

Note 2: We could also allow $\boldsymbol{\beta}_i = \boldsymbol{\beta}_{i,t}$ (e.g. estimate $\boldsymbol{\beta}_i$ over rolling windows for each t)

Factor Risk Budgeting

- Additively decompose (slice and dice) individual asset or portfolio return risk measures into factor contributions

- Allow portfolio manager to know sources of factor risk for allocation and hedging purposes

- Allow risk manager to evaluate portfolio from factor risk perspective

Factor Risk Decompositions

Assume asset or portfolio return R_t can be explained by a factor model

$$R_t = \alpha + \boldsymbol{\beta}'\mathbf{f}_t + \varepsilon_t$$
$$\mathbf{f}_t \sim iid\ (\boldsymbol{\mu}_f, \boldsymbol{\Omega}_f),\ \varepsilon_t \sim iid\ (0, \sigma_\varepsilon^2),\ cov(f_{k,t}, \varepsilon_s) = 0 \text{ for all } k, t, s$$

Re-write the factor model as

$$\begin{aligned} R_t &= \alpha + \boldsymbol{\beta}'\mathbf{f}_t + \varepsilon_t = \alpha + \boldsymbol{\beta}'\mathbf{f}_t + \sigma_\varepsilon \times z_t \\ &= \alpha + \tilde{\boldsymbol{\beta}}'\tilde{\mathbf{f}}_t \\ \tilde{\boldsymbol{\beta}} &= (\boldsymbol{\beta}', \sigma_\varepsilon)',\ \tilde{\mathbf{f}}_t = (\mathbf{f}_t, z_t)',\ z_t = \frac{\varepsilon_t}{\sigma_\varepsilon} \sim iid\ (0, 1) \end{aligned}$$

Then

$$\sigma_{FM}^2 = \tilde{\boldsymbol{\beta}}'\boldsymbol{\Omega}_{\tilde{f}}\tilde{\boldsymbol{\beta}},\ \boldsymbol{\Omega}_{\tilde{f}} = \begin{pmatrix} \boldsymbol{\Omega}_f & 0 \\ 0 & 1 \end{pmatrix}$$

Linearly Homogenous Risk Functions

Let $RM(\tilde{\boldsymbol{\beta}})$ denote the risk measures σ_{FM}, VaR_{α}^{FM} and ES_{α}^{FM} as functions of $\tilde{\boldsymbol{\beta}}$

Result 1: $RM(\tilde{\boldsymbol{\beta}})$ is a linearly homogenous function of $\tilde{\boldsymbol{\beta}}$ for $RM = \sigma_{FM}$, VaR_{α}^{FM} and ES_{α}^{FM}. That is, $RM(c\cdot\tilde{\boldsymbol{\beta}}) = c\cdot RM(\tilde{\boldsymbol{\beta}})$ for any constant $c \geq 0$

Example: Consider $RM(\tilde{\boldsymbol{\beta}}) = \sigma_{FM}(\tilde{\boldsymbol{\beta}})$. Then

$$\begin{aligned}
\sigma_{FM}(c \cdot \tilde{\boldsymbol{\beta}}) &= \left(c \cdot \tilde{\boldsymbol{\beta}}' \boldsymbol{\Omega}_{\tilde{f}} c \cdot \tilde{\boldsymbol{\beta}}\right)^{1/2} = c \cdot \left(\tilde{\boldsymbol{\beta}}' \boldsymbol{\Omega}_{\tilde{f}} \tilde{\boldsymbol{\beta}}\right)^{1/2} \\
&= c \cdot \sigma_{FM}(\tilde{\boldsymbol{\beta}})
\end{aligned}$$

Euler's Theorem and Additive Risk Decompositions

Result 2: Because $RM(\tilde{\boldsymbol{\beta}})$ is a linearly homogenous function of $\tilde{\boldsymbol{\beta}}$, by Euler's Theorem

$$
\begin{aligned}
RM(\tilde{\boldsymbol{\beta}}) &= \sum_{j=1}^{k+1} \tilde{\beta}_j \frac{\partial RM(\tilde{\boldsymbol{\beta}})}{\partial \tilde{\beta}_j} \\
&= \tilde{\beta}_1 \frac{\partial RM(\tilde{\boldsymbol{\beta}})}{\partial \tilde{\beta}_1} + \cdots + \tilde{\beta}_{k+1} \frac{\partial RM(\tilde{\boldsymbol{\beta}})}{\partial \tilde{\beta}_{k+1}} \\
&= \beta_1 \frac{\partial RM(\tilde{\boldsymbol{\beta}})}{\partial \beta_1} + \cdots + \beta_k \frac{\partial RM(\tilde{\boldsymbol{\beta}})}{\partial \beta_k} + \sigma_\varepsilon \frac{\partial RM(\tilde{\boldsymbol{\beta}})}{\partial \sigma_\varepsilon}
\end{aligned}
$$

Terminology

Factor j *marginal contribution to risk*

$$\frac{\partial RM(\tilde{\boldsymbol{\beta}})}{\partial \tilde{\beta}_j}$$

Factor j *contribution to risk*

$$\tilde{\beta}_j \frac{\partial RM(\tilde{\boldsymbol{\beta}})}{\partial \tilde{\beta}_j}$$

Factor j *percent contribution to risk*

$$\frac{\tilde{\beta}_j \frac{\partial RM(\tilde{\boldsymbol{\beta}})}{\partial \tilde{\beta}_j}}{RM(\tilde{\boldsymbol{\beta}})}$$

Analytic Results for $RM(\tilde{\boldsymbol{\beta}}) = \sigma_{FM}(\tilde{\boldsymbol{\beta}})$

$$\begin{aligned} \sigma_{FM}(\tilde{\boldsymbol{\beta}}) &= \left(\tilde{\boldsymbol{\beta}}'\boldsymbol{\Omega}_{\tilde{f}}\tilde{\boldsymbol{\beta}}\right)^{1/2} \\ \frac{\partial \sigma_{FM}(\tilde{\boldsymbol{\beta}})}{\partial \tilde{\boldsymbol{\beta}}} &= \frac{1}{\sigma_{FM}(\tilde{\boldsymbol{\beta}})}\boldsymbol{\Omega}_{\tilde{f}}\tilde{\boldsymbol{\beta}} \end{aligned}$$

Factor $j = 1, \ldots, K$ percent contribution to $\sigma_{FM}(\tilde{\boldsymbol{\beta}})$

$$\frac{\beta_1\beta_j cov(f_{1t}, f_{jt}) + \cdots + \beta_j^2 var(f_{jt}) + \cdots + \beta_K\beta_j cov(f_{Kt}, f_{jt})}{\sigma_{FM}^2(\tilde{\boldsymbol{\beta}})},$$

Asset specific factor contribution to risk

$$\frac{\sigma_\varepsilon^2}{\sigma_{FM}^2(\tilde{\boldsymbol{\beta}})},\ j = K + 1$$

Results for $RM(\tilde{\boldsymbol{\beta}}) = VaR_\alpha^{FM}(\tilde{\boldsymbol{\beta}})$, $ES_\alpha^{FM}(\tilde{\boldsymbol{\beta}})$

Based on arguments in Scaillet (2002), Meucci (2007) showed that

$$\begin{aligned}\frac{\partial VaR_\alpha^{FM}(\tilde{\boldsymbol{\beta}})}{\partial \tilde{\beta}_j} &= E[\tilde{f}_{jt}|R_t = VaR_\alpha^{FM}(\tilde{\boldsymbol{\beta}})],\ j = 1, \ldots, K+1 \\ \frac{\partial ES_\alpha^{FM}(\tilde{\boldsymbol{\beta}})}{\partial \tilde{\beta}_j} &= E[\tilde{f}_{jt}|R_t \leq VaR_\alpha^{FM}(\tilde{\boldsymbol{\beta}})],\ j = 1, \ldots, K+1\end{aligned}$$

Remarks

- Intuitive interpretation as stress loss scenario
- Analytic results are available under normality

Marginal Contributions to Tail Risk: Non-Parametric Estimates

Assume R_t and $\tilde{\mathbf{f}}_t$ are iid but make no distributional assumptions:

$$\{(R_1, \tilde{\mathbf{f}}_1), \ldots, (R_T, \tilde{\mathbf{f}}_T)\} = \text{observed iid sample}$$

Estimate marginal contributions to risk using *historical simulation*

$$\begin{aligned}
\hat{E}^{HS}[\tilde{f}_{jt}|R_t &= VaR_\alpha] = \\
&\frac{1}{m}\sum_{t=1}^{T} \tilde{f}_{jt} \cdot 1\left\{\widehat{VaR}_\alpha^{HS} - \varepsilon \leq R_t \leq \widehat{VaR}_\alpha^{HS} + \varepsilon\right\} \\
\hat{E}^{HS}[\tilde{f}_{jt}|R_t &\leq VaR_\alpha] = \frac{1}{[T\alpha]}\sum_{t=1}^{T} \tilde{f}_{jt} \cdot 1\left\{\widehat{VaR}_\alpha^{HS} \leq R_t\right\}
\end{aligned}$$

Problem: Not reliable with small samples or with unequal histories for R_t

Simulating Returns: Factor Model Monte Carlo

Assume asset or portfolio return R_{it} can be explained by a factor model

$$R_{it} = \alpha_i + \boldsymbol{\beta}_i' \mathbf{f}_t + \varepsilon_{it}$$

$$\mathbf{f}_t \sim iid\ (\boldsymbol{\mu}_f, \boldsymbol{\Omega}_f),\ \varepsilon_{it} \sim iid\ (0, \sigma^2_{\varepsilon,i}),\ cov(f_{k,t}, \varepsilon_{is}) = 0 \text{ for all } i, k, t, s$$

To simulate returns R_t

- Simulate from the pdf of $\mathbf{f}_t$
- Simulate from the pdf of ε_{it} (independent of $\mathbf{f}_t$)

This method is often called Factor Model Monte Carlo (FMMC)

Advantages of FMMC

- Number of factors is typically much smaller than the number of assets (e.g. 5 factors vs. 1000 assets)

- Multivariate modeling of f_t is feasible with a small number of factors

- Univariate models can be used for residuals ε_{it} because of independence across assets

- Dependence structure across assets is defined by factor loadings and dependence structure of factors

- Can deal with unequal histories for asset returns (e.g. hedge fund data)

Short History for Returns but Long History for Factors

$$\begin{array}{cccc} f_{1T} & \cdots & f_{KT} & R_{iT} \\ \vdots & \vdots & \vdots & \vdots \\ f_{1,T-T_i+1} & \cdots & f_{1,T-T_i+1} & R_{i,T-T_i+1} \\ \vdots & \vdots & \vdots & NA \\ f_{11} & \cdots & f_{1K} & NA \end{array}$$

- Observe full history for factors $\{\mathbf{f}_1, \ldots, \mathbf{f}_T\}$

- Observe partial history for assets (monotone missing data)

$$\begin{aligned} & \{R_{i,T-T_i+1}, \ldots, R_{iT}\}, \\ i &= 1, \ldots, n;\ t = T - T_i + 1, \ldots, T \end{aligned}$$

Simulation Algorithm

- Estimate factor models for each asset using partial history for assets and risk factors

$$R_{it} = \hat{\alpha}_i + \hat{\boldsymbol{\beta}}_i' \mathbf{f}_t + \hat{\varepsilon}_{it},\ t = T - T_i + \mathbf{1}, \ldots, T$$

- Simulate B values of the risk factors from the pdf of $\mathbf{f}_t$:

$$\{\mathbf{f}_1^*, \ldots, \mathbf{f}_B^*\}$$

- Simulate B values of the factor model residuals from the pdf of ε_{it}

$$\{\hat{\varepsilon}_{i1}^*, \ldots, \hat{\varepsilon}_{iB}^*\}$$

- Create pseudo factor model returns from fitted factor model parameters, simulated factor variables and simulated residuals:

$$\{R_1^*, \ldots, R_B^*\}$$
$$R_{it}^* = \hat{\boldsymbol{\beta}}_i' \mathbf{f}_t^* + \hat{\varepsilon}_{it}^*, \ t = 1, \ldots, B$$

Simulating Factor Realizations: Distribution choices

- Multivariate distributions (e.g., multivariate normal, t, copula distributions etc) (parametric, unconditional)

- Conditional multivariate distributions (e.g. normal DCC model)

- Empirical distribution (non-parametric, unconditional)
  - Resample with replacement from observed history of factors

- Filtered historical simulation (semi-parametric, conditional)

– use local time-varying factor covariance matrices to standardize factors prior to re-sampling and then re-transform with covariance matrices after re-sampling

Simulating Residuals: Distribution choices

- Normal distribution (parametric, unconditional)
- Non-normal: Student's t, Skewed Student's t etc. (parametric, unconditional)
- Empirical (resample with replacement from observed residuals) (nonparametric, unconditional)
- GARCH(1,1) (parametric, conditional)
- Filtered historical simulation (semi-parametric, conditional)

References

[1] Goldberg, L.R., Yayes, M.Y., Menchero, J., and Mitra, I. (2009). "Extreme Risk Analysis," MSCI Barra Research.

[2] Goldberg, L.R., Yayes, M.Y., Menchero, J., and Mitra, I. (2009). "Extreme Risk Management," MSCI Barra Research.

[3] Goodworth, T. and C. Jones (2007). "Factor-based, Non-parametric Risk Measurement Framework for Hedge Funds and Fund-of-Funds," The European Journal of Finance.

[4] Jiang, Y. (~~200~~ 2007). Overcoming Data Challenges in Fund-of-Funds Portfolio Management. PhD Thesis, Department of Statistics, University of Washington.

[5] Meucci, A. (2007). "Risk Contributions from Generic User-Defined Factors", Risk.