Financial Econometrics and Volatility Models
Estimating Realized Variance

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Outline

- Volatility Signature Plots
- Realized Variance and Market Microstructure Noise
- Unbiased Estimation of Realized Variance
- Empirical Applications
- Realized Covariance Estimation
- Bipower Variation and Tests for Jumps

Reading

- Bandi, F. and J. Russell (2006). "Separating Microstucture Noise from Volatility", *Journal of Financial Economics*, 79, 655-692
- Bandi, F. and J. Russell (2008). "Microstructure Noise, Realized Variance, and Optimal Sampling. *Review of Financial Studies*, 79, 339-369.
- Hansen, P.R. and A. Lunde (2006). "Realized Variance and Market Microstructure Noise," *Journal of Business and Economic Statistics*, 24(2), 127-161.

- Barndorff-Nielsen, O.E. and N. Shephard (2006). "Econometrics of Testing for Jumps in Financial Economics Using Bipower Variation," *Journal of Financial Econometrics*, 4(1), 1-50.

Volatility Signature Plots

- Observed log price (transaction price or mid-quote) for day $t = 1, \ldots, n$ is denoted $\tilde{p}_t$

- Divide each day into M subperiods and define $\delta = 1/M$. This creates a regularly spaced time clock

- Align observed prices to time clock
  - Previous tick method
  - Linear interpolation between adjacent ticks

- Define the jth inter-daily return for day t

$$\tilde{r}_{j,t} = \tilde{p}_{(t-1)+j\delta} - \tilde{p}_{(t-1)+(j-1)\delta}, \ j = 1, \ldots, M$$

- Define realized variance (RV) for day t at frequency M

$$RV_t^{(M)} = \sum_{j=1}^{M} \tilde{r}_{j,t}^2$$

Result: If returns are free of microstructure noise and prices follow a continuous time diffusion then

$$RV_t^{(M)} \xrightarrow{p} IV_t = \int_{t-1}^{t} \sigma^2(s)ds$$

Result: If M is chosen too large (δ is too small) then $RV_t^{(M)}$ becomes biased due to microstructure noise.

- Lack of liquidity could cause observed price to differ from true price (e.g. large trades or short time periods)

- Bid-Ask spread and discrete nature of price price data that implies rounding errors

- Econometric method to construct price data (infer prices from transaction data or mid-quotes; impute prices at times when no prices are observed)

- Data recording errors

Variance Signature Plots can be used to uncover biases due to microstructure noise

- Plot average realized variance, $\overline{RV}^{(M)} \equiv \frac{1}{n}\sum_{t=1}^{n} RV_t^{(M)}$, against the sampling frequency M, where the average is taken over n days

Example: Hansen and Lund (2004)

5 years of intra-day data for Alcoa and Microsoft $\quad (n = 1250)$

Compute $RV_t^{(M)}$ for $M = 1, \ldots, 3600$ seconds using transactions and mid-quotes and prices aligned using previous tick and linear interpolation methods.

Realized Variance and Market Microstructure Noise

Bandi, F. and J. Russell (2006). "Separating Microstucture Noise from Volatility", *Journal of Financial Economics*, 79, 655-692

Bandi, F. and J. Russell (2008). "Microstructure Noise, Realized Variance, and Optimal Sampling. *Review of Financial Studies*, 79, 339-369.

Main Points

- Observed HF log price = log efficient price + microstructure noise
  - Variance of daily return = variance of efficient returns + variance of microstructure noise

- Both unobserved components of variance can be estimated using HF data sampled at different frequencies
  - High frequency sampling captures microstructure noise
  - Low frequency sampling captures efficient return variance
- Provide procedure to purge HF return data of microstructure components and extract information on efficient return variance by sampling at optimal frequencies

Price Formation Mechanism

- Consider $t = 1, \ldots, n$ trading days.

- Observed log price at time t

$$\begin{aligned} \tilde{p}_t &= p_t + \eta_t \\ p_t &= \text{unobserved efficient price} \\ \eta_t &= \text{unobserved microstructure noise} \end{aligned}$$

- Divide each day into M subperiods and define $\delta = 1/M$. Define the jth inter-daily return for day t

$$\tilde{r}_{j,t} = \tilde{p}_{(t-1)+j\delta} - \tilde{p}_{(t-1)+(j-1)\delta},\ j = 1, \ldots, M$$

- Return decomposition

$$
\begin{aligned}
\tilde{r}_{j,t} &= r_{j,t} + \varepsilon_{j,t} \\
r_{j,t} &= p_{(t-1)+j\delta} - p_{(t-1)+(j-1)\delta} : \text{ efficient return} \\
\varepsilon_{j,t} &= \eta_{(t-1)+j\delta} - \eta_{(t-1)+(j-1)\delta} : \text{ microstructure noise}
\end{aligned}
$$

Assumption 1 (Efficient Price Process)

1. p_t is is a continuous stochastic volatility local martingale. Specifically,

$$p_t = m_t = \int_0^t \sigma_s dW_s, \; W_t = \text{Wiener process}$$

2. Spot volatility σ_t is cadlag and bounded away from zero

3. σ_t is independent of W_t for all t

4. Integrated volatility: $IV_t = \int_0^t \sigma_s^2 ds$

5. Quarticity $Q_t = \int_0^t \sigma_s^4 ds < M \leq \infty$

Assumption 2 (Microstructure Noise)

1. The random shocks η_t are iid, $E[\eta_t] = 0$, $E[\eta_t^2] = \sigma_\eta^2$, $E[\eta_t^8] < \infty$

2. True return process $r_{j,t}$ is independent of $\eta_{j,t}$ for all t and j

Remarks

- σ_s can display jumps, diurnal effects, high persistence (long memory) and nonstationarities

- $\varepsilon_{j,t}$ follows an MA(1) process independent of δ, with negative first order autocovariance

$$\begin{aligned} E[\varepsilon_{j,t}^2] &= 2E[\eta_{j,t}^2] = 2\sigma_\eta^2 \\ E[\varepsilon_{j,t}\varepsilon_{j-1,t}] &= -E[\eta_{j,t}^2] = -\sigma_\eta^2, E[\varepsilon_{j,t}\varepsilon_{j-k,t}] = 0,\ k > 1 \end{aligned}$$

  justified by Roll's bid-ask bounce model

- Efficient returns $r_{j,t}$ are of order $O_p(\sqrt{\delta})$ over periods of size δ

- Microstructure noise returns $\varepsilon_{j,t}$ are $O_p(1)$ over any time period

  – Price discreteness, bid-ask spread

- Longer period intra-day returns are less contaminated by noise than shorter period returns

Identification at High Frequencies: Volatility of the Unobserved Microstructure Noise

Squared Return Decomposition

$$\begin{aligned}\sum_{j=1}^{M} \tilde{r}_{j,t}^2 &= \sum_{j=1}^{M} r_{j,t}^2 + \sum_{j=1}^{M} \varepsilon_{j,t}^2 + 2\sum_{j=1}^{M} r_{j,t}\varepsilon_{j,t} \\ &= O_p(\sqrt{\delta}) + O_p(1) + O_p\left(\frac{1}{2}\right)\end{aligned}$$

Result (Bandi and Russell, 2004). As $M \to \infty$

$$\frac{1}{M}\sum_{j=1}^{M} \tilde{r}_{j,t}^2 \xrightarrow{p} E[\varepsilon^2],\ \frac{2}{M}\sum_{j=1}^{M} \tilde{r}_{j,t}^2 \xrightarrow{p} E[\eta^2]$$

Note: if η are iid across days $(t = 1, \ldots, n)$

$$\frac{1}{nM} \sum_{t=1}^{n} \sum_{j=1}^{M} \tilde{r}_{j,t}^{2} \xrightarrow{p} E[\varepsilon^2]$$

Use highest possible sample frequency to construct estimates

Identification at Low Frequencies: Volatility of the Unobserved Efficient Return

Result (Bandi and Russell, 2004). As $M \to \infty$

$$\sum_{j=1}^{M} r_{j,t}^2 \xrightarrow{p} \int_{t-1}^{t} \sigma_s^2 ds = IV_t, \quad \sum_{j=1}^{M} \varepsilon_{j,t}^2 \xrightarrow{p} \infty$$

$$\sum_{j=1}^{M} r_{j,t}\varepsilon_{j,t} = O_p(1)$$

Consequently,

$$\sum_{j=1}^{M} \tilde{r}_{j,t}^2 \to \infty$$

Hence, traditional RV estimator is inconsistent in the presence of microstructure noise!

Optimal Sampling: Balancing Bias-Variance Tradeoff

Intuition: RV estimator is expected to be less biased when sampled at low frequencies, since noise plays less of a roll when δ is large, but considerably more volatile. The optimal sampling frequency minimizes the MSE.

Result (Bandi and Russell, 2004)

$$
\begin{aligned}
\mathrm{MSE}\left(\sum_{j=1}^{M}\tilde{r}_{j,t}^{2}, IV_t\right) &= E_\sigma\left[\left(\sum_{j=1}^{M}\tilde{r}_{j,t}^{2} - IV_t\right)^2\right] \\
&= 2\frac{1}{M}\left(Q_t + o(1)\right) + M\beta + M^2\alpha + \gamma \\
\alpha &= \left(E[\varepsilon^2]\right)^2,\ \beta = 2E[\varepsilon^4] - 3\alpha \\
\gamma &= 4E[\varepsilon^2]IV_t - E[\varepsilon^4] + 2\alpha
\end{aligned}
$$

Remarks:

- The necessary indegrediants to compute the minimum of the MSE are

$$E[\varepsilon^2],\ E[\varepsilon^4] \text{ and } Q_t$$

- Consistent estimators for $E[\varepsilon^2]$ and $E[\varepsilon^4]$ (as $M \to \infty$)

$$\frac{1}{nM}\sum_{t=1}^{n}\sum_{j=1}^{M}\tilde{r}_{j,t}^2 \xrightarrow{p} E[\varepsilon^2],\ \frac{1}{nM}\sum_{t=1}^{n}\sum_{j=1}^{M}\tilde{r}_{j,t}^4 \xrightarrow{p} E[\varepsilon^4]$$

- Under microstructure noise, the BNS estimator of Q_t is inconsistent

$$\hat{Q}_t = \frac{M}{3}\sum_{j=1}^{M}\tilde{r}_{j,t}^4 \to \infty \text{ as } M \to \infty$$

- Bandi and Russell suggest to estimate Q_t using $\hat{Q}_t$ with a low sampling frequency (e.g. 15 minutes)

$$\hat{Q}_t = \frac{M^{low}}{3} \sum_{j=1}^{M^{low}} \tilde{r}_{j,t}^4$$

Result (Approximate optimal sampling frequency). The approximate optimal sampling frequency is chosen as the value $\delta_t^* = 1/M_t^*$ with

$$\begin{aligned} M_t^* &= \left(\frac{\hat{Q}_t}{\hat{\alpha}}\right)^{1/3} \\ \hat{\alpha} &= \left(\frac{1}{nM^{high}}\sum_{t=1}^{n}\sum_{j=1}^{M^{high}} \tilde{r}_{j,t}^2\right)^2, \ M \text{ is highest frequency} \\ \hat{Q}_t &= \frac{M^{low}}{3}\sum_{j=1}^{M^{low}} \tilde{r}_{j,t}^4, \ M^{low} \text{ is low frequency (15 mins)} \end{aligned}$$

Optimal sampling frequency RV estimate

$$RV_t^{(M_t^*)} = \sum_{j=1}^{M_t^*} \tilde{r}_{j,t}^2$$

Empirical Applications

- S&P 100 Stocks: 1993 - 2003
- Use mid-quotes as observed prices
- Compute optimal sampling frequencies for 100 stocks
  - Mean value of 4 minutes
  - Vary considerably over time

Hansen, P.R. and A. Lunde (2006). "Realized Variance and Market Microstructure Noise," Journal of Business and Economic Statistics, 24(2), 127-161.

Main points:

- Characterize how RV is affected by market microstructure noise under a general specification for the noise that allows for various forms of stochastic dependencies

- Market microstructure noise is time-dependent and correlated with efficient returns

- For Dow 30 stocks, noise may be ignored when returns are sampled a low frequencies (e.g. 20 mins)

Ugly Facts about Market Microstructure Noise

- Noise is correlated with efficient price
- Noise is time dependent
- Noise is quite small in Dow Jones 30 stocks
- Properties of noise have changed substantially over time

Notation and Assumptions

- $p^*(t) = p(t) + u(t) =$ efficient price $+$ noise
- $dp^*(t) = \sigma(t)dW(t)$
- Data are observed on interval $[a, b]$ (e.g. trading day)
- $IV = \int_a^b \sigma^2(t)dt$

- Partition $[a, b]$ into m subintervals

- ith subinterval is $[t_{i-1,m}, t_{i,m}]$

$$
\begin{aligned}
[a &= t_{0,m} < t_{1,m} < \cdots < t_{m,m} = b] \\
\delta_{i,m} &= t_{i,m} - t_{i-1,m}
\end{aligned}
$$

- Intra-day returns

$$
\begin{aligned}
y_{i,m} &= p(t_{i,m}) - p(t_{i-1,m}), i = 1, \ldots, m \\
&= y^*_{im} + e_{i,m} \\
y^*_{i,m} &= p^*(t_{i,m}) - p^*(t_{i-1,m}) \\
e_{i,m} &= u(t_{i,m}) - u(t_{i-1,m})
\end{aligned}
$$

- IV over $[t_{i-1,m}, t_{i,m}]$

$$\sigma^2_{i,m} = \int_{t_{i-1,m}}^{t_{i,m}} \sigma^2(s)ds = \mathsf{var}(y^*_{i,m})$$

- RV of efficient price

$$RV_*^{(m)} = \sum_{i=1}^{m} y^{*2}_{i,m}$$

- RV of observed price

$$RV^{(m)} = \sum_{i=1}^{m} y^2_{i,m}$$

Sampling Schemes

- Calendar time sampling (CTS)
  - Align prices to common regularly spaced time clock associated with $[a = t_{0,m} < t_{1,m} < \cdots < t_{m,m} = b]$
- Tick time sampling (TTS)
  - $t_{i,m}$ denotes actual transaction time
  - e.g. sample every fifth transaction

Characterizing the Bias of RV

Assumption 2: The noise process, u, is covariance stationary with mean 0, such that its autocovariance function is defined by $\pi(s) = E[u(t)u(t+s)]$

Remark: Assumption 2 allows for dependence between p^* and u

Decomposition of $RV^{(m)}$ when $y_{i,m} = y^*_{im} + e_{i,m}$

$$RV^{(m)} = \sum_{i=1}^{m} y^{*2}_{i,m} + 2\sum_{i=1}^{m} e_{i,m} y^*_{i,m} + \sum_{i=1}^{m} e^2_{i,m}$$

Theorem 1. Given Assumptions 1 and 2, the bias of $RV^{(m)}$ under CTS is given by

$$
\begin{aligned}
E[RV^{(m)} - IV] &= 2\rho_m + 2m\left[\pi(0) - \pi\left(\frac{b-a}{m}\right)\right] \\
\rho_m &= E\left[\sum_{i=1}^{m} e_{i,m} y_{i,m}^{*}\right]
\end{aligned}
$$

Remarks

- Bias always positive when $\mathrm{cov}(y_{im}^{*}, e_{i,m}) = 0$
- Bias can be negative when $\mathrm{cov}(y_{im}^{*}, e_{i,m}) < 0$ and large

Bias Corrected RV

Assumption 4: The noise process has finite dependence in the sense that $\pi(s) = 0$ for all $s > \theta_0$ for some $\theta_0 > 0$, and $E[u(t)|p^*(s)] = 0$ for all $|t-s| > \theta_0$

Theorem 2. Suppose Assumptions 1,2, and 4 hold and let q_m be such that $q_m/m > \theta_0$. Then under CTS,

$$E[RV^{(m)}_{AC_{q_m}} - IV] = 0$$

where

$$\begin{aligned} RV^{(m)}_{AC_{q_m}} &= \sum_{i=1}^{m} y^2_{i,m} + 2\sum_{h=1}^{q_m} \tilde{\gamma}_h \\ \tilde{\gamma}_h &= \frac{m}{m-h}\sum_{i=1}^{m-h} y_{i,m} y_{i+h,m} \end{aligned}$$

Remarks

- $RV^{(m)}_{AC_{q_m}}$ may be negative because $\tilde{\gamma}_h$ is not scaled downward in a way that would guarantee positivity (e.g. as with the NW type long-run variance estimator)

- One could use various kernels (e.g. Bartlett) to ensure positivity but the resulting estimators may not be unbiased

- $RV^{(m)}_{AC_{q_m}} \xrightarrow{p} IV$ as $m \to \infty$ because $q_m/m \to 0$ sufficiently fast

Realized Kernel Estimators

Barndorff-Nielsen, O.E., P. Hansen, A. Lunde, N. Shephard (2008). "Designing realised kernels to measure the ex-post variation of equity prices in the presence of noise," Unpublished Working Paper (http://ssrn.com/abstract=620203)

Idea: Create approximately unbiased RV estimators that have good finite sample properties

$$
\begin{aligned}
RV^{(m)}_{\mathsf{ker}_{q_m}} &= \sum_{i=1}^{m} y_{i,m}^2 + 2\sum_{h=1}^{q_m} k\left(\frac{h-1}{q_m}\right)\tilde{\gamma}_h \\
\tilde{\gamma}_h &= \frac{m}{m-h}\sum_{i=1}^{m-h} y_{i,m}y_{i+h,m} \\
k(\cdot) &= \text{kernel weight function} \\
q_m &= \text{lag truncation parameter}
\end{aligned}
$$

Realized Covariance Estimation

Griffin, J.E. and R.C.A. Oomen (2006). “Covariance Measurement in the Presence of Non-Synchronous Trading and Market Microstructure Noise,” Unpublished paper, Department of Statistics, University of Warwick.

Payseur, S. (2008). “Essays on Realized Covariance Estimation,” Phd thesis, University of Washington.

Payseur, S. (2008). Realized Software Package.

Notation

- Observed vector of log prices for k assets, aligned to a common time clock equally spaced by $\delta = 1/M$, for day $t = 1, \dots, n$ is denoted $\tilde{\mathbf{p}}_t = (p_t^1, \dots, p_t^2)'$.

- Define the jth inter-daily return vector for day t

$$\tilde{\mathbf{r}}_{j,t} = \tilde{\mathbf{p}}_{(t-1)+j\delta} - \tilde{\mathbf{p}}_{(t-1)+(j-1)\delta},\ j = 1, \dots, M$$

- Define $k \times k$ realized covariance (RCOV) matrix for day t at frequency M

$$\mathbf{RCOV}_t^{(M)} = \sum_{j=1}^{M} \tilde{r}_{j,t} \tilde{r}'_{j,t}$$

- For 2 assets with intra-day returns $r^1_{j,t}$ and $r^2_{j,t}$ define

$$RCOV_t^{(M)} = \sum_{j=1}^{M} \tilde{r}^1_{j,t} \tilde{r}^2_{j,t}$$

Result: If returns are free of microstructure noise and prices follow a continuous time diffusion then

$$\mathbf{RCOV}_t^{(M)} \xrightarrow{p} \mathbf{ICOV}_t = \int_{t-1}^{t} \mathbf{\Sigma}(t)dt$$

Result: If M is chosen too large (δ is too small) then $\mathbf{RCOV}_t^{(M)}$ becomes biased due to microstructure noise. However, the bias of the off diagonal terms is different from the bias observed in realized variance due to non-synchronous trading.

Covariance Signature Plots

Average Plot: Plot average pairwise realized covariance variance, $\overline{RCOV}^{(M)} \equiv \frac{1}{n}\sum_{t=1}^{n} RCOV_t^{(M)}$, against the sampling frequency M, where the average is taken over n days.

Remark: Payseur (2008) argues that averaging over days masks the instability of $RCOV_t^{(M)}$ for different M, and advocates the use of 1-day pairwise covariance signature plots.

Kernel Estimators for Realized Covariance

$$
\begin{aligned}
\mathbf{RCOV}^{(M)}_{t,\mathsf{ker}_{q_m}} &= \sum_{j=1}^{M} r_{j,t} r'_{j,t} + 2 \sum_{h=1}^{q_m} k\left(\frac{h-1}{q_m}\right) (\tilde{\Gamma}_h + \tilde{\Gamma}'_h) \\
\tilde{\Gamma}_h &= \frac{M}{M-h} \sum_{j=1}^{M-h} r_{j,t} r'_{j+h,t} \\
k(\cdot) &= \text{kernel weight function} \\
q_m &= \text{lag truncation parameter}
\end{aligned}
$$

Realized Bipower Variation and Tests for Jumps

Barndorff-Nielsen, O.E. and N. Shephard (2004). "Power and Bipower Variation with Stochastic Volatility and Jumps," *Journal of Financial Econometrics*, 2, 1-48.

Barndorff-Nielsen, O.E. and N. Shephard (2006). "Econometrics of Testing for Jumps in Financial Economics Using Bipower Variation," *Journal of Financial Econometrics*, 4(1), 1-50.

Main point: Show how realized bipower variation can be used to create test statistics to construct nonparametric tests for the presence of jumps in the price process.

Notation and Quadratic Variation

- $p_t =$ continuous time log price process (semi-martingale)

$$\begin{aligned} p_t &= p_t^c + p_t^d, \ t > 0 \\ p_t^c &= \text{continuous part} \\ p_t^d &= \text{discontinuous (jump) part} \end{aligned}$$

- Quadratic variation defined

$$\begin{aligned} QV_t &= p - \lim_{n \to \infty} \sum_{j=0}^{n-1} (p_{t_{j+1}} - p_{t_j})^2 \\ &= QV_t^c + QV_t^d \\ QV_t^d &= \sum_{0 \le u < t} \Delta Y_u^2, \ \Delta Y_t = Y_t - Y_{t-} = \text{jumps} \end{aligned}$$

- Intra-day returns for $\delta = 1/m$

$$r_j = p_{j\delta} - p_{(j-1)\delta}, \ j = 1, \ldots, t/\delta$$

- Realized quadratic variation (realized variance)

$$RV_t^{(m)} = \sum_{j=1}^{[t/\delta]} y_j^2$$

Assume p_t belongs to the Brownian semimartingale plus jump (BSMJ) process

$$\begin{aligned} p_t &= \int_0^t a_s ds + \int_0^t \sigma_s dW_s + Z_t \\ Z_t &= \sum_{j=1}^{N_t} c_j \end{aligned}$$

$N_t =$ counting process and c_j are non-zero random variables.

Result:

$$\begin{aligned} QV_t &= QV_t^c + QV_t^d \\ QV_t^c &= \int_0^t \sigma_s^2 ds \\ QV_t^d &= \sum_{j=1}^{N_t} c_j^2 \end{aligned}$$

Bipower Variation

Defn: The 1,1-order BPV process is

$$BPV_t^{1,1} = p - \lim_{\delta \to 0} \sum_{j=2}^{[t/\delta]} |y_{j-1}||y_j|$$

Result: If $p_t \in$ BSMJ with $a = 0$ and σ independent of W then

$$\begin{aligned} BPV_t^{1,1} &= \mu_1^2 \int_0^t \sigma_s^2 ds = \mu_1^2 QV_t^c \\ \mu_1 &= E[z] = \sqrt{2}/\sqrt{\pi},\ z \sim N(0,1) \end{aligned}$$

Hence,

$$\mu_1^{-2} BPV_t^{1,1} = QV_t^c$$

Result:

$$QV_t - \mu_1^{-2} BPV_t^{1,1} = QV_t^d = \sum_{j=1}^{N_t} c_j^2$$

Jump component can be estimated using realized variance and realized bipower variation

$$\begin{aligned} QV_t^d &= RV_t^{(m)} - \mu_1^{-2} RBPV_t^{1,1(m)} \\ RBPV_t^{1,1(m)} &= \sum_{j=2}^{[t/\delta]} |y_{j-1}||y_j| \end{aligned}$$

Testing for Jumps

Theorem 1. Let p_t belong to the Brownian semimartingale (BSM) process without jumps, and suppose that σ_s is independent of W. Then, as $\delta \longrightarrow 0$

$$
\begin{aligned}
G &= \frac{\delta^{-1/2}\left(\mu_1^{-2} RBPV_t^{1,1(m)} - RV_t^{(m)}\right)}{\left(\theta \int_0^t \sigma_u^4 du\right)^{1/2}} \longrightarrow N(0,1) \\
H &= \frac{\delta^{-1/2}\left(\frac{\mu_1^{-2} RBPV_t^{1,1(m)}}{RV_t^{(m)}} - 1\right)}{\left(\theta \frac{\int_0^t \sigma_u^4 du}{\left\{\int_0^t \sigma_u^2 du\right\}^2}\right)^{1/2}} \longrightarrow N(0,1)
\end{aligned}
$$

where $\theta = (\pi^2/4) + \pi - 5$.

Feasible Tests

Problem: Need a consistent estimator of $\int_0^t \sigma_u^4 du$ under null and alternative

Solution: Use realized quadpower variation

$$RBPV_t^{1,1,1,1(m)} = \sum_{j=4}^{[t/\delta]} |y_{j-3}||y_{j-2}||y_{j-1}||y_j|$$

Result:

$$RBPV_t^{1,1,1,1(m)} \to \mu_1^4 \int_0^t \sigma_u^4 du$$

$$\hat{G} = \frac{\delta^{-1/2}\left(\mu_1^{-2} RBPV_t^{1,1(m)} - RV_t^{(m)}\right)}{\left(\theta \mu_1^{-4} RBPV_t^{1,1,1,1(m)}\right)^{1/2}} \rightarrow N(0,1)$$

$$\hat{H} = \frac{\delta^{-1/2}\left(\frac{\mu_1^{-2} RBPV_t^{1,1(m)}}{RV_t^{(m)}} - 1\right)}{\left(\theta \frac{RBPV_t^{1,1,1,1(m)}}{\left\{RBPV_t^{1,1(m)}\right\}^2}\right)^{1/2}} \rightarrow N(0,1)$$