Amath 546/Econ 589
Estimating Macroeconomic Factor Models for Asset Returns

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Outline

1. Macroeconomic factor models

2. Sharpe's single factor model

3. General multiple factor model

Three Types of Factor Models

1. Macroeconomic factor model

   (a) Factors are observable economic and financial time series

2. Fundamental factor model

   (a) Factors are created from observerable asset characteristics

3. Statistical factor model

   (a) Factors are unobservable and extracted from asset returns

Macroeconomic Factor Models

$$
\begin{aligned}
R_{it} &= \alpha_i + \boldsymbol{\beta}_i' \mathbf{f}_t + \varepsilon_{it} \\
\mathbf{f}_t &= \text{observed economic/financial time series}
\end{aligned}
$$

Econometric problems:

- Choice of factors
- Estimate factor betas, $\boldsymbol{\beta}_i$, and residual variances, σ_i^2, using time series regression techniques.
- Estimate factor covariance matrix, $\boldsymbol{\Omega}_f$, from observed history of factors

Shape's Single Factor Model

Sharpe's single factor model is a macroeconomic factor model with a single market factor:

$$R_{it} = \alpha_i + \beta_i R_{Mt} + \varepsilon_{it},\ i = 1, \ldots, N; t = 1, \ldots, T \tag{1}$$

where R_{Mt} denotes the return or excess return (relative to the risk-free rate) on a market index (typically a value weighted index like the S&P 500 index) in time period t.

Risk-adusted expected return and abnormal return

$$\begin{aligned} E[R_{it}] &= \beta_i E[R_{Mt}] \\ \alpha_i &= E[R_{it}] - \beta_i E[R_{Mt}] \end{aligned}$$

Covariance matrix of assets

$$\mathbf{\Omega}_{FM} = \sigma_M^2 \boldsymbol{\beta}\boldsymbol{\beta}' + \mathbf{D} \tag{2}$$

where

$$\begin{aligned}
\sigma_M^2 &= var(R_{Mt}) \\
\boldsymbol{\beta} &= (\beta_1, \ldots, \beta_N)' \\
\mathbf{D} &= diag(\sigma_1^2, \ldots, \sigma_N^2), \\
\sigma_i^2 &= var(\varepsilon_{it})
\end{aligned}$$

Estimation

Because R_{Mt} is observable, the parameters β_i and σ_i^2 of the single factor model (1) for each asset can be estimated using time series regression (i.e., ordinary least squares) giving

$$
\begin{aligned}
\mathbf{R}_i &= \widehat{\alpha}_i \mathbf{1}_T + \mathbf{R}_M \widehat{\beta}_i + \widehat{\boldsymbol{\varepsilon}}_i, \ i = 1, \ldots, N \\
\widehat{\beta}_i &= \widehat{cov}(R_{it}, R_{Mt}) / \widehat{var}(R_{Mt}) = \hat{\sigma}_{iM} / \hat{\sigma}_M^2 \\
\widehat{\alpha}_i &= \bar{R}_i - \hat{\beta}_i \bar{R}_M \\
\widehat{\sigma}_i^2 &= \frac{1}{T-2} \widehat{\boldsymbol{\varepsilon}}_i' \widehat{\boldsymbol{\varepsilon}}_i
\end{aligned}
$$

The estimated single factor model covariance matrix is

$$
\widehat{\mathbf{\Omega}}_{FM} = \widehat{\sigma}_M^2 \widehat{\boldsymbol{\beta}} \widehat{\boldsymbol{\beta}}' + \widehat{\mathbf{D}}
$$

Remarks

1. Computational efficiency may be obtained by using multivariate regression. The coefficients α_i and β_i and the residual variances σ_i^2 may be computed in one step in the multivariate regression model

$$\mathbf{R} = \mathbf{X}\mathbf{\Gamma}' + \mathbf{E}$$

The multivariate OLS estimator of $\mathbf{\Gamma}'$ is

$$\widehat{\mathbf{\Gamma}}' = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{R}'.$$

The estimate of the residual covariance matrix is

$$\widehat{\mathbf{\Sigma}} = \frac{1}{T-2}\widehat{\mathbf{E}}'\widehat{\mathbf{E}}$$

where $\hat{\mathbf{E}} = \mathbf{R} - \mathbf{X}\hat{\mathbf{\Gamma}}'$ is the multivariate least squares residual matrix. The diagonal elements of $\widehat{\mathbf{\Sigma}}$ are the diagonal elements of $\widehat{\mathbf{D}}$.

2. The R^2 from the time series regression is a measure of the proportion of "market" risk, and $1 - R^2$ is a measure of asset specific risk. Additionally, $\widehat{\sigma}_i$ is a measure of the typical size of asset specific risk. Given the variance decomposition

$$var(R_{it}) = \beta_i^2 var(R_{Mt}) + var(\varepsilon_{it}) = \beta_i^2 \sigma_M^2 + \sigma_i^2$$

R^2 can be estimated using

$$R^2 = \frac{\hat{\beta}_i^2 \hat{\sigma}_M^2}{\widehat{var}(R_{it})}$$

3. The single factor covariance matrix (2) is constant over time. This may not be a good assumption. There are several ways to allow (2) to vary over time. In general, β_i, σ_i^2 and σ_M^2 can be time varying. That is,

$$\beta_i = \beta_{it},\ \sigma_i^2 = \sigma_{it}^2,\ \sigma_M^2 = \sigma_{Mt}^2.$$

To capture time varying betas, rolling regression or Kalman filter techniques could be used. To capture conditional heteroskedasticity, GARCH models may be used for σ_{it}^2 and σ_{Mt}^2. One may also use exponential weights in computing estimates of β_{it}, σ_{it}^2 and σ_{Mt}^2. A time varying factor model covariance matrix is

$$\widehat{\mathbf{\Omega}}_{FM,t} = \hat{\sigma}_{Mt}^2 \widehat{\boldsymbol{\beta}}_t \widehat{\boldsymbol{\beta}}_t' + \widehat{\mathbf{D}}_t,$$

General Multi-factor Model

Model specifies K observable macro-variables

$$R_{it} = \alpha_i + \boldsymbol{\beta}_i' \mathbf{f}_t + \varepsilon_{it}$$

- Chen, Roll and Ross (1986) provides a description of commonly used macroeconomic factors for equity based on the APT. Lo (2008) discusses factor models for hedge funds.

- Sometimes the macroeconomic factors are standardized to have mean zero and a common scale.

- The factors must be *stationary* (not trending).

- Sometimes the factors are made orthogonal.

Estimation

Because the factor realizations are observable, the parameter matrices $\mathbf{B}$ and $\mathbf{D}$ of the model may be estimated using time series regression:

$$
\begin{aligned}
\mathbf{R}_i &= \widehat{\alpha}_i \mathbf{1}_T + \mathbf{F}\widehat{\boldsymbol{\beta}}_i + \widehat{\boldsymbol{\varepsilon}}_i = \mathbf{X}\hat{\gamma} + \widehat{\boldsymbol{\varepsilon}}_i,\ i = 1, \ldots, N \\
\mathbf{X} &= [\mathbf{1}_T \vdots \mathbf{F}],\ \hat{\gamma} = (\hat{\alpha}_i, \hat{\boldsymbol{\beta}}_i')' = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{R}_i \\
\widehat{\sigma}_i^2 &= \frac{1}{T - K - 1}\widehat{\boldsymbol{\varepsilon}}_i'\widehat{\boldsymbol{\varepsilon}}_i
\end{aligned}
$$

The covariance matrix of the factor realizations may be estimated using the time series sample covariance matrix

$$
\widehat{\boldsymbol{\Omega}}_f = \frac{1}{T-1}\sum_{t=1}^{T}(\mathbf{f}_t - \overline{\mathbf{f}})(\mathbf{f}_t - \overline{\mathbf{f}})',\ \overline{\mathbf{f}} = \frac{1}{T}\sum_{t=1}^{T}\mathbf{f}_t
$$

The estimated multifactor model covariance matrix is then

$$
\widehat{\boldsymbol{\Omega}}_{FM} = \widehat{\mathbf{B}}\widehat{\boldsymbol{\Omega}}_f\widehat{\mathbf{B}}' + \widehat{\mathbf{D}} \tag{3}
$$

Remarks

1. $\boldsymbol{\Omega}_{FM}$ can be made time varying by allowing $\boldsymbol{\beta}_i$, $\boldsymbol{\Omega}_f$ and σ_i^2 $(i = 1, \ldots, N)$ to be time varying

Example: Estimation of Single Index Model in R using investment data from Berndt (1991).