Amath 546/Econ 589
Estimating Fundamental Factor Models for Asset Returns

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Factor models for asset returns have the general form

$$\begin{aligned} R_{it} &= \alpha_i + \beta_{1i} f_{1t} + \beta_{2i} f_{2t} + \cdots + \beta_{Ki} f_{Kt} + \varepsilon_{it} \\ &= \alpha_i + \boldsymbol{\beta}_i' \mathbf{f}_t + \varepsilon_{it} \end{aligned}$$

- R_{it} is the simple return (real or in excess of the risk-free rate) on asset i $(i = 1, \ldots, N)$ in time period t $(t = 1, \ldots, T)$,

- f_{kt} is the k^{th} *common factor* $(k = 1, \ldots, K)$,

- β_{ki} is the *factor loading* or *factor beta* for asset i on the k^{th} factor,

- ε_{it} is the asset *specific factor*.

Recall, the factor model may be rewritten as a *cross-sectional* regression model at time t by stacking the equations for each asset to give

$$\begin{aligned}
\underset{(N\times 1)}{\mathbf{R}_t} &= \underset{(N\times 1)}{\boldsymbol{\alpha}} + \underset{(N\times K)}{\mathbf{B}}\underset{(K\times 1)}{\mathbf{f}_t} + \underset{(N\times 1)}{\boldsymbol{\varepsilon}_t}, \; t = 1, \ldots, T \\
\underset{(N\times K)}{\mathbf{B}} &= \begin{bmatrix} \boldsymbol{\beta}_1' \\ \vdots \\ \boldsymbol{\beta}_N' \end{bmatrix} = \begin{bmatrix} \beta_{11} & \cdots & \beta_{1K} \\ \vdots & \ddots & \vdots \\ \beta_{N1} & \cdots & \beta_{NK} \end{bmatrix} \\
E[\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t' | \mathbf{f}_t] &= \mathbf{D} = diag(\sigma_1^2, \ldots, \sigma_N^2)
\end{aligned}$$

Fundamental Factor Models

Fundamental factor models use observable asset specific characteristics (fundamentals) like industry classification, market capitalization, style classification (value, growth) etc. to determine the common risk factors.

- Factor betas are constructed from observable asset characteristics (i.e., $\mathbf{B}$ is known at time t)

- Factor realizations, $\mathbf{f}_t$, are estimated/constructed for each t given $\mathbf{B}$

- In practice, fundamental factor models are estimated in two ways.

BARRA Approach

- This approach was pioneered by Bar Rosenberg, founder of BARRA Inc., and is discussed at length in Grinold and Kahn (2000), Conner et al (2010), Cariño et al (2010), and various Barra research reports (available at `www.barra.com`).

- In this approach, the observable asset specific fundamentals (or some transformation of them) are treated as the factor betas, $\boldsymbol{\beta}_i$. Some of these are time invariant (e.g. industry classification) and some are time varying (e.g. market capitalization)

- The factor realizations at time t, $\mathbf{f}_t$, are unobserved. The econometric problem is then to estimate the factor realizations at time t given the factor betas. This is done by running a cross-section regression at time t. To get the time series of factor returns, T cross-section regressions must be run.

BARRA Integrated Model (BIM)

- Multi-asset class factor model covering
  - Equity (single country models)
  - Fixed income (single country models)
  - Alternatives (currencies, commodities, hedge funds, private real estate)
- Large set of local factors for individual models are mapped into a small set of global factors for integrated model

Barra US Equity Factor Model

$$
\begin{aligned}
R_{it} &= \alpha_i + \boldsymbol{\beta}'_{1i}\mathbf{f}_{1t} + \boldsymbol{\beta}'_{2i}\mathbf{f}_{2t} + \varepsilon_{it} \\
\mathbf{f}_{1t} &= \text{55 industry factors} \\
\mathbf{f}_{2t} &= \text{12 ``style'' factors}
\end{aligned}
$$

Style Factors	
Volatility	Earnings yield
Momentum	Value
Size	Earnings variation
Size nonlinearity	Leverage
Trading activity	Currency sensitivity
Growth	Dividend yield

Fama-French Approach

- This approach was introduced by Eugene Fama and Kenneth French (1992).

- For a given observed asset specific characteristic, e.g. size, they determined factor realizations using a two step process. First they sorted the cross-section of assets based on the values of the asset specific characteristic. Then they formed a hedge portfolio which is long in the top quintile of the sorted assets and short in the bottom quintile of the sorted assets. The observed return on this hedge portfolio at time t is the observed factor realization for the asset specific characteristic. This process is repeated for each asset specific characteristic.

- Given the observed factor realizations for $t = \mathbf{1}, \ldots, T$, the factor betas for each asset are estimated using N time series regressions.

Fama-French Three Factor Model

$$
\begin{aligned}
R_{it} &= \alpha_i + \beta_1 f_{1t} + \beta_2 f_{2t} + \beta_3 f_{3t} + \varepsilon_t \\
&= \alpha_i + \beta_M R_{Mt} + \beta_{SMB} R_{SMB,t} + \beta_{HML} R_{HML,t} + \varepsilon_{it} \\
R_{Mt} &= \text{excess return on market index (market factor)} \\
R_{SMB,t} &= \text{excess return on ``Small minus Big'' portfolio (size factor)} \\
R_{HML,t} &= \text{excess return on ``High minus Low'' portfolio (value factor)}
\end{aligned}
$$

Note: FF factors can be downloaded from Ken French's website at Dartmouth University.

BARRA-type Single Factor Model

Consider a single factor model in the form of a cross-sectional regression at time t

$$\underset{(N\times1)}{\mathbf{R}_t} = \underset{(N\times1)}{\boldsymbol{\beta}}\underset{(1\times1)}{f_t} + \underset{(N\times1)}{\boldsymbol{\varepsilon}_t}, t = 1, \ldots, T$$

- $\boldsymbol{\beta}$ is an $N \times 1$ vector of observed values of an asset specific attribute (e.g., market capitalization, industry classification, style classification)

- f_t is an unobserved factor realization.

- $var(f_t) = \sigma_f^2$; $cov(f_t, \varepsilon_{it}) = 0$, for all i, t; $var(\varepsilon_{it}) = \sigma_i^2, i = 1, \ldots, N$.

Remarks

- If attribute k represents industry classification, then $\beta_{k,i} = 1$ if asset i is in industry; 0 otherwise

- If attribute k represents "size", for example, then $\beta_{k,i}$ is typically a $z-$score type variable constructed by sorting all stocks by size (e.g. market capitalization) and then standardizing the sorted data

$$\beta_{k,i} = \frac{\text{market cap}_i - \text{mean(market cap)}}{\text{sd(market cap)}}$$

  Then $\beta_{k,i} > 2$ indicates a very large firm; $\beta_{k,i} < -2$ indicates a very small firm

Estimation

For each time period $t = 1, \ldots T$, the vector of factor betas, $\boldsymbol{\beta}$, is treated as data and the factor realization f_t, is the parameter to be estimated. Since the error term $\boldsymbol{\varepsilon}_t$ is heteroskedastic, efficient estimation of f_t is done by weighted least squares (WLS) (assuming the asset specific variances σ_i^2 are known)

$$\begin{aligned} \hat{f}_{t,wls} &= (\boldsymbol{\beta}'\mathbf{D}^{-1}\boldsymbol{\beta})^{-1}\boldsymbol{\beta}'\mathbf{D}^{-1}\mathbf{R}_t,\ t = 1, \ldots, T \\ \mathbf{D} &= diag(\sigma_1^2, \ldots, \sigma_N^2) \end{aligned} \tag{1}$$

Note 1: σ_i^2 can be consistently estimated and a feasible WLS estimate can be computed

$$\begin{aligned} \hat{f}_{t,fwls} &= (\boldsymbol{\beta}'\hat{\mathbf{D}}^{-1}\boldsymbol{\beta})^{-1}\boldsymbol{\beta}'\hat{\mathbf{D}}^{-1}\mathbf{R}_t,\ t = 1, \ldots, T \\ \hat{\mathbf{D}} &= diag(\hat{\sigma}_1^2, \ldots, \hat{\sigma}_N^2) \end{aligned}$$

Note 2: Other weights besides $\hat{\sigma}_i^2$ could be used; e.g., market capitalization weights

Factor Mimicking Portfolio

The WLS estimate of f_t in (1) has an interesting interpretation as the return on a portfolio $\mathbf{h} = (h_1, \ldots, h_N)'$ that solves

$$\min_{\mathbf{h}} \frac{1}{2}\mathbf{h}'\mathbf{D}\mathbf{h} \text{ subject to } \mathbf{h}'\boldsymbol{\beta} = 1$$

The portfolio $\mathbf{h}$ minimizes asset return residual variance subject to having unit exposure to the attribute $\boldsymbol{\beta}$ and is given by

$$\mathbf{h}' = (\boldsymbol{\beta}'\mathbf{D}^{-1}\boldsymbol{\beta})^{-1}\boldsymbol{\beta}'\mathbf{D}^{-1}$$

The estimated factor realization is then the portfolio return

$$\hat{f}_{t,wls} = \mathbf{h}'\mathbf{R}_t$$

When the portfolio $\mathbf{h}$ is normalized such that $\sum_i^N h_i = 1$, it is referred to as a *factor mimicking portfolio*.

BARRA-type Industry Factor Model

Consider a stylized BARRA-type industry factor model with K mutually exclusive industries. The factor sensitivities β_{ik} in (**??**) for each asset are time invariant and of the form

$$\begin{aligned} \beta_{ik} &= 1 \text{ if asset } i \text{ is in industry } k \\ &= 0, \text{ otherwise} \end{aligned}$$

and f_{kt} represents the factor realization for the k^{th} industry in time period t.

- The factor betas are dummy variables indicating whether a given asset is in a particular industry.

- The estimated value of f_{kt} will be equal to the weighted average excess return in time period t of the firms operating in industry k.

Industry Factor Model Regression

The industry factor model with K industries is summarized as

$$\begin{aligned} R_{it} &= \beta_{i1}f_{1t} + \cdots + \beta_{iK}f_{Kt} + \varepsilon_{it},\ i = 1, \ldots, N; t = 1, \ldots, T \\ \text{var}(\varepsilon_{it}) &= \sigma_i^2,\ i = 1, \ldots, N \\ \text{cov}(\varepsilon_{it}, f_{jt}) &= 0,\ j = 1, \ldots, K;\ i = 1, \ldots, N \\ \text{cov}(f_{it}, f_{jt}) &= \sigma_{ij}^f,\ i, j = 1, \ldots, K \end{aligned}$$

where

$$\begin{aligned} \beta_{ik} &= 1 \text{ if asset } i \text{ is in industry k } (k = 1, \ldots, K) \\ &= 0,\ \text{otherwise} \end{aligned}$$

It is assumed that there are N_k firms in the kth industry such $\sum_{k=1}^K N_k = N$.

Estimation of Industry Factor Model Factors

Consider the cross-section regression at time t

$$\begin{aligned}
\mathbf{R}_t &= \boldsymbol{\beta}_1 f_{1t} + \cdots + \boldsymbol{\beta}_K f_{Kt} + \boldsymbol{\varepsilon}_t, \\
&= \mathbf{B}\mathbf{f}_t + \boldsymbol{\varepsilon}_t \\
E[\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t'] &= \mathbf{D}, \text{ cov}(\mathbf{f}_t) = \boldsymbol{\Omega}_f
\end{aligned}$$

Since the industries are mutually exclusive it follows that

$$\boldsymbol{\beta}_j' \boldsymbol{\beta}_k = N_k \text{ for } j = k, \ 0 \text{ otherwise}$$

An unbiased but inefficient estimate of the factor realizations $\mathbf{f}_t$ can be obtained by OLS:

$$\widehat{\mathbf{f}}_{t,\mathsf{OLS}} = (\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}'\mathbf{R}_t = \begin{pmatrix} \widehat{f}_{1t,\mathsf{OLS}} \\ \vdots \\ \widehat{f}_{Kt,\mathsf{OLS}} \end{pmatrix} = \begin{pmatrix} \frac{1}{N_1}\sum_{i=1}^{N_1} R_{it}^1 \\ \vdots \\ \frac{1}{N_K}\sum_{i=1}^{N_K} R_{it}^K \end{pmatrix}$$

Estimation of Factor Realization Covariance Matrix

Given $(\widehat{\mathbf{f}}_{1,\mathsf{OLS}}, \ldots, \widehat{\mathbf{f}}_{T,\mathsf{OLS}})$, the covariance matrix of the industry factors may be computed as the time series sample covariance

$$\begin{aligned}
\widehat{\mathbf{\Omega}}^F_{\mathsf{OLS}} &= \frac{1}{T-1}\sum_{t=1}^{T}(\widehat{\mathbf{f}}_{t,\mathsf{OLS}} - \bar{\mathbf{f}}_{\mathsf{OLS}})(\widehat{\mathbf{f}}_{t,\mathsf{OLS}} - \bar{\mathbf{f}}_{\mathsf{OLS}})', \\
\bar{\mathbf{f}}_{\mathsf{OLS}} &= \frac{1}{T}\sum_{t=1}^{T}\widehat{\mathbf{f}}_{t,\mathsf{OLS}}
\end{aligned}$$

Estimation of Residual Variances

The residual variances, $\text{var}(\varepsilon_{it}) = \sigma_i^2$, can be estimated from the time series of residuals from the T cross-section regressions as follows. Let $\hat{\boldsymbol{\varepsilon}}_{t,\text{OLS}}$, $t = 1, \ldots, T$, denote the $(N \times 1)$ vector of OLS residuals, and let $\hat{\varepsilon}_{it,\text{OLS}}$ denote the i^{th} row of $\hat{\boldsymbol{\varepsilon}}_{t,\text{OLS}}$. Then σ_i^2 may be estimated using

$$
\begin{aligned}
\hat{\sigma}^2_{i,\text{OLS}} &= \frac{1}{T-1}\sum_{t=1}^{T}(\hat{\varepsilon}_{it,\text{OLS}} - \bar{\varepsilon}_{i,\text{OLS}})^2,\ i = 1, \ldots, N \\
\bar{\varepsilon}_{i,\text{OLS}} &= \frac{1}{T}\sum_{t=1}^{T}\hat{\varepsilon}_{it,\text{OLS}}
\end{aligned}
$$

Estimation of Industry Factor Model Asset Return Covariance Matrix

The covariance matrix of the N assets is estimated using

$$\widehat{\mathbf{\Omega}}_{\mathsf{OLS}} = \mathbf{B}\widehat{\mathbf{\Omega}}^{F}_{\mathsf{OLS}}\mathbf{B}' + \widehat{\mathbf{D}}_{\mathsf{OLS}}$$

where $\widehat{\mathbf{D}}_{\mathsf{OLS}}$ is a diagonal matrix with $\widehat{\sigma}^2_{i,\mathsf{OLS}}$ along the diagonal.

Weighted Least Squares Estimation

- The OLS estimation of the factor realizations $\mathbf{f}_t$ is inefficient due to the cross-sectional heteroskedasticity in the asset returns.

- The estimates of the residual variances may be used as weights for weighted least squares (feasible GLS) estimation:

$$
\begin{aligned}
\widehat{\mathbf{f}}_{t,\mathsf{GLS}} &= (\mathbf{B}'\widehat{\mathbf{D}}_{\mathsf{OLS}}^{-1}\mathbf{B})^{-1}\mathbf{B}'\widehat{\mathbf{D}}_{\mathsf{OLS}}^{-1}\mathbf{R}_t,\ t = 1, \ldots, T \\
\widehat{\mathbf{\Omega}}_{\mathsf{GLS}}^{F} &= \frac{1}{T-1}\sum_{t=1}^{T}(\widehat{\mathbf{f}}_{t,\mathsf{GLS}} - \overline{\mathbf{f}}_{\mathsf{GLS}})(\widehat{\mathbf{f}}_{t,\mathsf{GLS}} - \overline{\mathbf{f}}_{\mathsf{GLS}})' \\
\widehat{\sigma}_{i,\mathsf{GLS}}^{2} &= \frac{1}{T-1}\sum_{t=1}^{T}(\widehat{\varepsilon}_{it,\mathsf{GLS}} - \overline{\varepsilon}_{i,\mathsf{GLS}})^2,\ i = 1, \ldots, N \\
\widehat{\mathbf{\Omega}}_{\mathsf{GLS}} &= \mathbf{B}\widehat{\mathbf{\Omega}}_{\mathsf{GLS}}^{F}\mathbf{B}' + \widehat{\mathbf{D}}_{\mathsf{GLS}}
\end{aligned}
$$

Example: Estimation of Industry Factor Model in R using investment data from Berndt (1991).