Econ 589: Financial Econometrics
HW 3

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Due: Tuesday 5/16/2011

1 Reading

1. Ding, Z., and Engle, R.F. (2001). "Large Scale Conditional Covariance Matrix Modeling, Estimation and Testing," unpublished manuscript, Department of Economics, UC San Diego (on class syllabus page)

2. Zivot, E., and Wang, J. (2006). *Modeling Financial Time Series with S-PLUS, Second Edition.* Chapter 13.

3. Jondeau, E., Poon, S.-H., and Rockinger, M. (2006), *Financial Modeling Under Non-Gaussian Distributions*, Chapters 4-6.

4. Taylor, S.J. (2005), *Asset Price Dynamics, Volatility, and Prediction.* Chapters 9 and 10.

2 Part I: Analytic Questions

2.1 Multivariate GARCH

Consider the following bivariate GARCH models for the time varying covariance matrix

$$\mathbf{\Sigma}_t = \begin{pmatrix} \sigma_t^{11} & \sigma_t^{12} \\ \sigma_t^{12} & \sigma_t^{22} \end{pmatrix}, \ \text{vech}(\mathbf{\Sigma}_t) = \begin{pmatrix} \sigma_t^{11} \\ \sigma_t^{12} \\ \sigma_t^{22} \end{pmatrix}$$

DVECH(1,1):

$$\begin{aligned} \underset{3\times1}{\mathbf{h}_t} &= \underset{3\times1}{\mathbf{a}_0} + \underset{3\times3}{\text{diag}(\mathbf{a}_1)}\underset{3\times1}{\mathbf{v}_{t-1}} + \underset{3\times3}{\text{diag}(\mathbf{b}_1)}\underset{3\times1}{\mathbf{h}_{t-1}} \\ \mathbf{h}_t &= \text{vech}(\mathbf{\Sigma}_t), \ \mathbf{v}_t = \text{vech}(\boldsymbol{\varepsilon}_{t-1}\boldsymbol{\varepsilon}_{t-1}') \\ \mathbf{a}_0 &= \text{vech}(\mathbf{A}_0), \ \mathbf{a}_1 = \text{vech}(\mathbf{A}_1), \ \mathbf{b}_1 = \text{vech}(\mathbf{B}_1) \end{aligned}$$

where $\mathbf{A}_0$, $\mathbf{A}_1$ and $\mathbf{B}_1$ are 2×2 symmetric matrices. Here, for $\mathbf{a}_1 = (a_1, a_2, a_3)'$

$$\text{diag}(\mathbf{a}_1) = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$$

Scalar MD(1,1):

$$\underset{2\times2}{\mathbf{\Sigma}_t} = \underset{2\times2}{\mathbf{A}_0\mathbf{A}_0'} + \underset{2\times2}{(\mathbf{a}_1\mathbf{a}_1')} \odot (\boldsymbol{\epsilon}_{t-1}\boldsymbol{\epsilon}_{t-1}') + \underset{2\times2}{(\mathbf{b}_1\mathbf{b}_1')} \odot \mathbf{\Sigma}_{t-j}$$

where $\mathbf{A}_0$ is a lower triangular matrix, $\mathbf{a}_i$ and $\mathbf{b}_i$ are are 2×1 vectors.

BEKK(1,1)

$$\mathbf{\Sigma}_t = \mathbf{A}_0\mathbf{A}_0' + \mathbf{A}_1(\boldsymbol{\epsilon}_{t-1}\boldsymbol{\epsilon}_{t-1}')\mathbf{A}_1' + \mathbf{B}_1\mathbf{\Sigma}_{t-1}\mathbf{B}_1'$$

where $\mathbf{A}_0$ is a 2×1 lower triangular matrix, but $\mathbf{A}_1$ and $\mathbf{B}_1$ are unrestricted 2×2 square matrices.

1. For each model, write out the equations for σ_t^{11} σ_t^{12} and σ_t^{22}. Briefly compare and contrast each model.

2. For each model, assuming stationarity derive the unconditional variance matrix $\bar{\mathbf{\Sigma}} = E[\mathbf{\Sigma}_t] = E[\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t']$.

3. For each model, derive the s-step ahead forecasting function for Σ_{t+s}.

2.2 Stochastic Volatility

Consider the log-normal AR(1) SV model

$$\begin{gathered} r_t = \sigma_t u_t = \exp(w_t/2)u_t, \ t = 1, \ldots, n \\ w_t = \ln \sigma_t^2 = \omega + \phi w_{t-1} + \eta_{w,t} \\ (u_t, \eta_{w,t})' \sim \text{iid } N(\mathbf{0}, \text{diag}(1, \sigma_{\eta_w}^2)) \\ \alpha_w = \frac{\omega}{1-\phi}, \ \beta_w^2 = \frac{\sigma_{\eta_w}^2}{1-\phi^2} \end{gathered}$$

For $0 < \phi < 1$ and $\sigma_{\eta_w} \geq 0$, the series r_t is strictly stationary and ergodic, and unconditional moments of all orders exist.

1. Using the properties of the SV model and the log-normal distribution, derive

the following moment conditions typically used for GMM estimation:

$$\begin{aligned}
E[|r_t|] &= (2/\pi)^{1/2}E[\sigma_t] \\
E[r_t^2] &= E[\sigma_t^2] \\
E[|r_t^3|] &= 2\sqrt{2/\pi}E[\sigma_t^3] \\
E[r_t^4] &= 3E[\sigma_t^4] \\
E[|r_t r_{t-j}|] &= (2/\pi)E[\sigma_t\sigma_{t-j}], \\
E[r_t^2 r_{t-j}^2] &= E[\sigma_t^2\sigma_{t-j}^2],
\end{aligned}$$

where for any positive integer j and positive constants p and s,

$$\begin{aligned}
E[\sigma_t^p] &= \exp\left(\frac{p\alpha_w}{2} + \frac{p^2\beta_w^2}{8}\right) \\
E[\sigma_t^p\sigma_{t-j}^s] &= E[\sigma_t^p]E[\sigma_t^s]\exp\left(\frac{ps\phi^j\beta_w^2}{4}\right)
\end{aligned}$$

2.3 Continuous time Models

For the Ito process

$$\begin{aligned}
dX(t) &= \mu(X(t),t)dt + \sigma(X(t),t)dW(t),\ W(t) = \text{ Wiener process} \\
&= \mu dt + \sigma dW(t),
\end{aligned}$$

Ito's Lemma is: For a continuous functional $G(X(t),t)$

$$dG(X(t),t) = \left(\frac{\partial G}{\partial x}\mu + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\sigma^2\right)dt + \frac{\partial G}{\partial x}\sigma dW(t)$$

1. Assume that the log price $p(t) = \ln P(t)$ follows the SDE

$$dp(t) = \gamma dt + \sigma dW(t).$$

   Using Ito's Lemma, derive the SDE for the price process $P(t) = \exp(p(t))$

2. Consider a forward price F of a nondividend-paying stock, we have

$$F(t,T) = P(t)e^{r_f(T-t)},$$

   where r_f = risk-free rate of interest, which is constant, and $P(t)$ is the current stock price. Suppose $P(t)$ follows the geometric Brownian motion

$$dP(t) = \mu P(t)dt + \sigma P(t)dW(t).$$

   Using Ito's Lemma, derive the SDE for $F(t,T)$.

3 Part II: Empirical Problems

Consider the daily returns on Microsoft and the S&P 500 that we have been using in the class examples. These are available in the S+FinMetrics module and are posted in Excel files on the class webpage (for use in other software programs). You may use any software you like (e.g. Eviews, Matlab, Ox, R, Stata, S-PLUS) but I recommend using either Eviews, R or S-PLUS.

Consider the log-normal AR(1) SV model for demeaned returns $y_t = r_t - \mu$:

$$r_t = \exp(w_t/2)u_t,$$
$$w_t = \omega + \phi w_{t-1} + \eta_{w,t}$$

Then

$$\begin{aligned} \ln r_t^2 &= w_t + \ln u_t^2 \\ E[\ln u_t^2] &= -1.27, \ \mathrm{var}(\ln u_t^2) = \pi^2/2 \end{aligned}$$

The linear (but non-Gaussian) state space model has measurement equation

$$\ln r_t^2 = -1.27 + w_t + \zeta_t, \ \zeta_t \sim iid \ (0, \pi^2/2)$$

and transition equation

$$w_t = \omega + \phi w_{t-1} + \eta_{w,t}, \ \eta_{w,t} \sim iid \ N(0, \sigma^2_{\eta_w})$$

The parameters to be estimated are $\theta = (\omega, \phi, \sigma^2_{\eta_w})'$. For numerical stability when $r_t^2 \approx 0$, it is recommended to use the transformation

$$\begin{aligned} r_t^2 &= \ln(r_t^2 + s) - \frac{s}{r_t^2 + s} \\ s &= \widehat{\mathrm{var}}(r_t) \times 0.02 \end{aligned}$$

1. For the MSFT and SP500 returns, estimate the SV model by QMLE. Be careful about your choice of starting values. From the properties of the log-normal SV model, you can use the method of moments to get starting values for ω and $\sigma^2_{\eta_w}$. A reasonable starting value for ϕ is 0.9.

2. Using your estimated results, use the Kalman filter prediction and updating equations to generate forecasts of $w_t = \ln(\sigma_t)$. Convert these forecasts to forecasts for σ_t.

Estimation Hints

- Estimation in Eviews. Eviews implements state space modeling. See the online help for details.

- Estimation in S+FinMetrics. S+FinMetrics 3.0 implements the SsfPack state space modeling and Kalman filtering tools developed by Siem Jan Koopman. See Zivot and Wang (2006) Chapter 14.

- Estimation in R. The R package dlm has state space modeling and Kalman filtering tools similar to the SsfPack tools in S+FinMetrics. See the online vignette for examples of how to set up a state space model and estimate it by QMLE.