Amath 546/Econ 589 Homework 2

Eric Zivot

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Question 1. Let R denote the simple daily return on an asset and assume that $R \sim N(0.01, 0.10)$. For the initial wealth $V_0 = 100$, the profit and loss are random variables defined as $\Pi = V_0 R$ and $L = -\Pi$. Let $\alpha \in (0.1)$ denote the confidence level for the daily VaR.

1. Derive the normal distributions for Π and L.

2. Give mathematical expressions for VaR_α based on the normal distributions for R, Π and L.

3. Using the expressions from the previous question, compute VaR_α for $\alpha = 0.95$ and 0.99. The values based on R, Π and L should all be equivalent.

Question 2. Let R denote the simple daily return on an asset and assume that $R \sim N(\mu, \sigma^2)$, where μ and σ^2 are unknown and must be estimated from an observed sample of size T. A natural estimate for daily VaR_α is

$$\widehat{VaR}_\alpha = -V_0 \hat{q}^R_{1-\alpha} \tag{1}$$

where

$$\hat{q}^R_{1-\alpha} = \hat{\mu} + \hat{\sigma} \times q^Z_{1-\alpha} \tag{2}$$

$\hat{\mu}$ is the sample mean, $\hat{\sigma}$ is the sample standard deviation and and $q^Z_{1-\alpha}$ the $1-\alpha$ lower quantile of $Z \sim N(0,1)$. The Central Limit Theorem gives the result

$$\begin{pmatrix} \hat{\mu} \\ \hat{\sigma} \end{pmatrix} \sim N\left(\begin{pmatrix} \mu \\ \sigma \end{pmatrix}, \begin{pmatrix} \frac{\sigma^2}{T} & 0 \\ 0 & \frac{\sigma^2}{2T} \end{pmatrix} \right) \tag{3}$$

which implies that, for large enough T, $\hat{\mu} \sim N\left(\mu, \frac{\sigma^2}{T}\right)$, $\hat{\sigma} \sim N\left(\sigma, \frac{\sigma^2}{2T}\right)$ and that $\hat{\mu}$ and $\hat{\sigma}$ are independent.

1. Use (1) - (3) to derive mathematical expressions for $var(\widehat{VaR}_\alpha)$ and $SE(\widehat{VaR}_\alpha) = \sqrt{var(\widehat{VaR}_\alpha)}$.

2. Assuming $\sigma = 0.10$ and $V_0 = 100$, plot $SE(\widehat{VaR}_\alpha)$ for a grid of 25 α values between $\alpha = 0.90$ and $\alpha = 0.995$ for $T = 25, 50$ and 100. How well is VaR_α estimated for α values close to 1?

Question 3. This problem is an example to illustrate that VaR is not subadditive and can discourage diversification. A company is selling par \$1000 bonds with a maturity of one year that pay a simple interest of 5% so that the bond pays \$50 at the end of one year if the company does not default. If the bank defaults, then the entire \$1000 is lost. The probability of no default is 0.96. To make the loss distribution continuous, assume that the loss is distributed $N(-50, 1)$ with probability 0.96 and $N(1000, 1)$ with probability 0.04[1]. Suppose that there is a second bond company selling bonds with exactly the same loss distribution and that the two companies are independent. Consider two portfolios. Portfolio 1 buys two bonds from the first company and portfolio 2 buys one bond from each of the two companies. Both portfolios have the same expected loss but portfolio 2 is more diversified.

1. Derive the loss CDFs for portfolio 1 and portfolio 2.

2. For $\alpha = 0.95$, compute VaR_α for portfolio 1 and portfolio 2.

3. Why does this example illustrate the subaddivitity of VaR?

Question 4. In this question you will use the PerformanceAnalytics package to estimate daily historical, normal and modified (Cornish-Fisher) VaR for Microsoft and the S&P 500 index. First, download daily adjusted closing prices on Microsoft (ticker MSFT) and the S&P 500 (ticker ^GSPC) over the period 2000-01-03 to 2012-04-03. Compute simple daily returns from both sets of prices. For automatically downloading data in R, you can use the `getSymbols()` function from the quantmod package or the `get.hist.quote()` function from the tseries package. For calculating returns you can use the `CalculateReturns()` function from the PerformanceAnalytics package.

1. Plot the daily returns on MSFT and the S&P 500. Note any of the stylized facts we discussed in class. Using the PerformanceAnalytics function `chart.Histogram()` plot the histograms with a normal curve overlaid, and using the PerformanceAnalytics `chart.QQPlot()` plot the normal QQ-plots. Does the normal distribution look appropriate for these two assets? Use the PerformanceAnalytics function `table.Stats()` to compute descriptive statistics for the returns on the two assets. Note the sample values of skewness and excess kurtosis.

[1] The main purpose of making the loss distribution continuos is to simplify calculations. However, the loss will be continuous, for example, if the portfolio contained both the bond and some stocks.

2. Using the PerformanceAnalytics function `VaR()`, estimate daily 95% and 99% VaR for the two assets based on the empirical distribution (i.e., historical VaR), the normal distribution and the Cornish-Fisher distribution (i.e., modified VaR). The `VaR()` function calculates VaR based on the distribution of returns and gives VaR_α as the lower $1-\alpha$-quantile of the return distribution. Summarize the results nicely in a table and comment. In addition, for each asset create a plot showing the returns together with horizontal lines indicating the 99% VaR values for the three methods. Note: be sure to read the online help for the function `VaR()`.

3. Using the PerformanceAnalytics function `ES()`, estimate daily 95% and 99% ES for the two assets based on the empirical distribution (i.e., historical ES), the normal distribution and the Cornish-Fisher distribution (i.e., modified ES). The `ES()` function calculates ES based on the distribution of returns and gives ES_α as the mean return less than the lower $1-\alpha$-quantile of the return distribution. Summarize the results nicely in a table and comment. In addition, for each asset create a plot showing the returns together with horizontal lines indicating the 99% ES values for the three methods. Note: be sure to read the online help for the function `ES()`.

Question 5. In this question you will use the PerformanceAnalytics functions StdDev, VaR and ES to estimate risk budgets based volatility, VaR and ES for an equally weighted portfolio of Microsoft and the S&P 500 index.

1. Using the PerformanceAnalytics function `StdDev()`, decompose the volatility of an equally weighted portfolio of MSFT and S&P 500 into the individual asset components. Which assets contributes most to the volatility of the portfolio?

2. Using the PerformanceAnalytics function `VaR()`, decompose the 95% Gaussian, historical and modified VaR of an equally weighted portfolio of MSFT and S&P 500 into the individual asset components. Summarize the results nicely in a table and comment. Which assets contributes most to the 95% VaR of the portfolio? Are the results similar to the decomposition based on the volatility?

3. Using the PerformanceAnalytics function `ES()`, decompose the 95% Gaussian, historical and modified ES of an equally weighted portfolio of MSFT and S&P 500 into the individual asset components. Summarize the results nicely in a table and comment. Which assets contributes most to the 95% ES of the portfolio? Are the results similar to the decomposition based on the VaR?