Econ 512 Lec 9

Note Title 4/26/2010

VaR

L = losses

R = returns, $L = -R \cdot W_0$

Multivariate GARCH: CCC model

$$\Sigma_t = \begin{bmatrix} \sigma_t^{11} & \sigma_t^{12} & \cdots & \sigma_t^{1k} \\ & \sigma_t^{22} & & \\ & & \ddots & \\ & & & \sigma_t^{kk} \end{bmatrix}$$

constant correlations

$$= \begin{bmatrix} (\sigma_t^{11})^{1/2} & & 0 \\ & \ddots & \\ 0 & & (\sigma_t^{kk})^{1/2} \end{bmatrix} * \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1k} \\ \vdots & \ddots & & \\ & & \ddots & \\ \rho_{1k} & & & \cdot \end{bmatrix}$$

$$\begin{bmatrix} (\sigma_t^{11})^{1/2} & & 0 \\ & \ddots & \\ 0 & & (\sigma_t^{kk})^{1/2} \end{bmatrix}$$

Univariate GARCH models

$$\sigma_t^{12} = \sigma_t^{11} \cdot \rho_{12}$$

time varying volatility.

time invariant correlation

time varying conditional covariance

Aside

Factor Models

$$r_{jt} = \beta_{j1} f_{1t} + \dots + \beta_{jM} f_{Mt} + u_t \qquad \begin{array}{l} t=1,\dots,T \\ j=1,\dots,K \\ m=1,\dots,M \end{array}$$

$m << k$ (e.g. $M = 2$ or 3)

f_{jt} = common factors $j=1,\dots,M$

$\sim GARCH(1,1)$

u_t = residual $\sim iid(0, \sigma_u^2)$

Method I: use observable portfolios as factors

e.g. CAPM

$$r_{jt} = \beta_j r_{mt} + u_{jt}$$

r_{mt} = return on mkt index.

$$Var(r_{jt} \mid I_{t-1}) = \beta_j^2 \cdot var(r_{mt} \mid I_{t-1})$$

$+ \text{var}(u_{it} \mid I_{t-1})$

$+ 2\,\text{cov}(r_{mt}, u_{it})$

Assume: $\text{var}(u_{it} \mid I_{t-1}) = \sigma_{iu}^2$ constant

$\text{cov}(r_{mt}, u_{it} \mid I_{t-1}) = 0$

β_i is constant as well

Method 2: Principle Components

$\underset{100\times 1}{\underset{\sim}{r_t}} \sim (0, \underset{100\times 100}{\Sigma})$, $\underset{\sim}{r_t} = (r_{1t}, \dots, r_{kt})'$

$\text{cov}(r_t) = \underset{100\times 100}{\Sigma} = \underset{100\times 100}{\Lambda} \underset{100\times 100}{\Delta} \underset{100\times 100}{\Lambda}$ = spectral decomposition

Δ = diagonal matrix with eigen-values (sorted from largest to smallest)

$$= \begin{bmatrix} \Delta_1 & & & 0 \\ & \Delta_2 & & \\ & & \ddots & \\ 0 & & & \Delta_{100} \end{bmatrix}$$

$$\Delta_1 > \Delta_2 > \cdots > \Delta_{100}$$

$$\underset{100\times100}{\Lambda} = [\underset{100\times1}{\underset{\sim}{\lambda_1}} \cdots \underset{100\times1}{\underset{\sim}{\lambda_{100}}}]$$ = orthogonal matrix of eigenvectors

$$\underset{\sim}{\lambda_1} = \begin{bmatrix} \lambda_{11} \\ \vdots \\ \lambda_{1,100} \end{bmatrix}$$

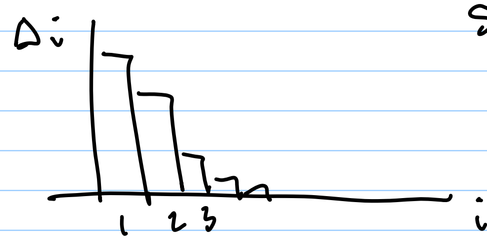
Λ orthogonal $\Rightarrow \Lambda = \Lambda' = \Lambda^{-1}$

$\Rightarrow \|\lambda_i\|^2 = 1$ and $\lambda_i' \lambda_j = 0$

λ_i = principle components $i = 1, \ldots, 100$

$$\Sigma = \Lambda \Delta \Lambda' = \sum_{i=1}^{100} \underset{\text{scalar}}{\Delta_i} \underset{k\times1}{\lambda_i} \underset{1\times k}{\lambda_i'}$$

If there is a factor structure with m common factors then exactly m eigenvalues will be non-zero and k-m eigenvalues will be zero



Scree plot

$$f_{1t} = \underset{1\times k}{\lambda_1'}\,\underset{k\times 1}{r_t} = \sum_{i=1}^{k} \lambda_{1i} \cdot r_{it}$$

$$f_{2t} = \underset{1\times k}{\lambda_2'}\,\underset{k\times 1}{r_t} = \sum_{i=1}^{k} \lambda_{2i}\, r_{it}$$

f_{1t} is uncorrelated with f_{2t}

~~1st~~ principle component $f_{1,t}$ solves

$$\max_{w} \; w'\Sigma w \quad \text{s.t.} \quad w'w = 1$$

2nd principle component

$$\max_{x} \; x'\Sigma x \quad \text{s.t.} \quad x'x = 1$$
$$\text{and } x'w = 0$$

$$\Sigma_t = A_0 + A_1 \odot \epsilon_{t-1}\epsilon_{t-1}' + h_1 \odot \Sigma_{t-1}$$

$$\epsilon_t = \Sigma_t^{1/2} z_t \;, \quad z_t \sim N(0, I_K]$$

$$\epsilon_t \epsilon_t' = \Sigma_t^{1/2} z_t z_t' \Sigma_t^{1/2\,\prime}$$

$$E\{\epsilon_t \epsilon_t' \mid I_{t-1}) = \Sigma_t^{1/2} E[z_t z_t' \mid I_{t-1}] \Sigma_t^{1/2}$$
$$= \Sigma_t^{1/2} \cdot I_K \Sigma_t^{1/2\,\prime}$$
$$= \Sigma_t$$

$r_t = \beta_1 f_{1t} + \cdots + \beta_m f_{mt} + u_t$

PCA estimate factors by eigen-vectors

$r_t = \beta_1 \hat{f}_{1t} + \cdots + \beta_m \hat{f}_{mt} + u_t$

$\hat{f}_{1t} = \hat{\Delta}_1' r_t$

estimated factor from principle components analysis

$E[\sigma_t^4] = ?$

$var(X) = E[X^2] - E[X]^2$

$X = \sigma_t^2$

$var(\sigma_t^2) = E[\sigma_t^4] - E[\sigma_t^2]^2$

$\Rightarrow E[\sigma_t^4] = var(\sigma_t^2) + E[\sigma_t^2]^2$