Econ 512 Lec 4

Note Title 4/7/2010

ARCH

$$\varepsilon_t = z_t \cdot \sigma_t \,, \quad z_t \sim (0,1) \qquad \text{ARCH(1)}$$

$$\sigma_t^2 = a_0 + a_1 \varepsilon_{t-1}^2 \qquad I_{t-1} = \{\varepsilon_{t-1}, \varepsilon_{t-2}, \ldots\}$$

$$E\{\varepsilon_t \mid I_{t-1}\} = E[z_t \sigma_t \mid I_{t-1}]$$

$$= \sigma_t \underbrace{E[z_t \mid I_{t-1}]}_{=0}$$

$$= 0$$

$$E\{\varepsilon_t\} = E\{E\{\varepsilon_t \mid I_{t-1}\}\}$$

$$= E\{0\} = 0$$

$$E[\varepsilon_t^2 \mid I_{t-1}] = E\{z_t^2 \sigma_t^2 \mid I_{t-1}\}$$

$$= \sigma_t^2 \underbrace{E\{z_t^2 \mid I_{t-1}\}}_{1}$$

$$= \sigma_t^2$$

$$E[\epsilon_t^2] = E[E[\epsilon_t^2 | I_{t-1}]]$$
$$= E[\sigma_t^2]$$
$$E[\epsilon_t \epsilon_{t-j} | I_{t-1}] = E[\epsilon_t \epsilon_{t-j}] = 0$$
$\{\epsilon_t, I_{t-1}\}$ is a MDS

$$\epsilon_t = z_t \cdot \sigma_t$$
$$\sigma_t^2 = a_0 + a_1 \epsilon_{t-1}^2$$
$$E[\sigma_t^2] = a_0 + a_1 E[\epsilon_{t-1}^2]$$
$$= a_0 + a_1 E[\sigma_{t-1}^2]$$
Assume σ_t^2 is stationary then
$$E[\sigma_t^2] = E[\sigma_{t-1}^2]$$
$$\Rightarrow E[\sigma_t^2] = a_0 + a_1 E[\sigma_t^2]$$

$$\Rightarrow E(\sigma_t^2) = \frac{a_0}{1-a_1} \qquad |a_1| < 1$$

$\bar{\sigma}^2$

Stationarity condition.

$$\sigma_t^2 = a_0 + a_1 \epsilon_{t-1}^2$$

$$\bar{\sigma}^2 = \frac{a_0}{1-a_1} \Rightarrow (1-a_1)\cdot\bar{\sigma}^2 = a_0$$

$$\sigma_t^2 = (1-a_1)\bar{\sigma}^2 + a_1 \epsilon_{t-1}^2$$

$$\Rightarrow \sigma_t^2 - \bar{\sigma}^2 = a_1(\epsilon_{t-1}^2 - \bar{\sigma}^2) \qquad |a_1|$$

$$\sigma_t^2 = a_0 + a_1 \epsilon_{t-1}^2$$

add ϵ_t^2 to both sides

$$\sigma_t^2 + \epsilon_t^2 = a_0 + a_1 \epsilon_{t-1}^2 + \epsilon_t^2$$

$$\Rightarrow \epsilon_t^2 = a_0 + a_1 \epsilon_{t-1}^2 + \epsilon_t^2 - \sigma_t^2$$

v_t

$$\epsilon_t^2 = a_0 + a_1 \epsilon_{t-1}^2 + v_t \quad : AR(1)$$

$$v_t = \epsilon_t^2 - \sigma_t^2 \text{ is MDS}$$

$$E\{v_t \mid I_{t-1}\} = 0$$

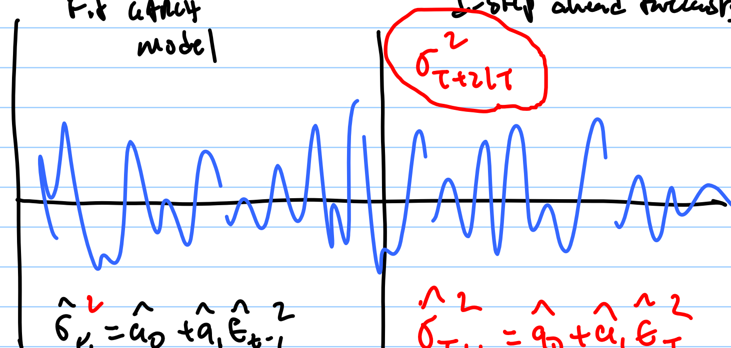
In Sample

Fit GARCH model

Out-of-sample

1-step ahead forecasts

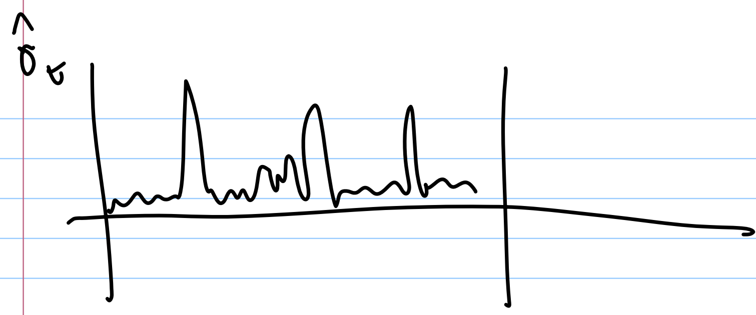
$\sigma^2_{T+2|T}$



$$\hat{\sigma}_t^2 = \hat{a}_0 + \hat{a}_1 \hat{\epsilon}_{t-1}^2 + \hat{b}_1 \hat{\sigma}_{t-1}$$

T

$$\hat{\sigma}_{T+1}^2 = \hat{a}_0 + \hat{a}_1 \hat{\epsilon}_T^2 + \hat{\sigma}_T^2$$


$\hat{\sigma}_t$