Financial Econometrics and Volatility Models
Continuous-Time Stochastic Processes

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Reading

- APDVP, chapters 13 and 14
- MFTS, chapter 20
- FMUND, chapters 13-17

Continuous Time Stochastic Processes

- A continuous-time continuous stochastic process is defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where Ω is a nonempty sample space, $\mathcal{F}$ is a $\sigma-$field consisting of subsets of Ω, and $\mathcal{P}$ is a probability measure. Such a process can be written as $\{X(\omega, t)\}$, where t denotes time and is continuous in $[0, \infty]$.

- For a given $t \in [0, \infty]$, $X(\omega, t)$ is a real-valued continuous random variable. For a given $\omega \in \Omega$, $\{X(\omega, t)\}$ is a time series with values depending on the time index t

- For simplicity, $X(\omega, t)$ is denoted $X(t)$ instead of X_t to emphasize that t is continuous

The Wiener Process

A Wiener process $W(t)$ (aka a standard Brownian motion process) is a continuous-time stochastic process with sample paths defined for $0 \leq t \leq T$ that satisfies the following properties:

- $W(0) = 0$
- $W(t) - W(s) \sim N(0, t - s)$ whenever $t > s$
- $W(v) - W(u)$ is independent of $W(t) - W(s)$ whenever $v > u \geq t > s$
- $W(t)$ is a continuous process - there are no jumps in its sample paths

Remarks

1. The independence of the increments $W(v) - W(u)$ and $W(t) - W(s)$ is a random walk property

2. A sample path of $W(t)$ is continuous in t, but not differentiable in t!

3. A discretized version of $W(t)$ can be generated from
$$W_t = W_{t-1} + \sqrt{\Delta t}\varepsilon_t, \; W_0 = 0, \; \varepsilon_t \sim N(0, 1)$$
Divide the interval $[0, t]$ into T increments of length Δt such that $t = T \cdot \Delta t$. Then
$$W_t = \sum_{j=1}^{T} \sqrt{\Delta t}\varepsilon_j \sim N(0, t)$$

Donsker's Theorem for Partial Sums (Functional Central Limit Theorem)

Let $\{\varepsilon_i\}_{i=1}^n$ be a sequence of iid $N(0,1)$ random variables. For any $r \in [0,1]$ let $[nr]$ denote the integer part of $n \cdot r$ and define the partial sum process

$$W_n(r) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nr]} \varepsilon_i$$

Note that $W_n(r)$ can be thought of as random function of r. Then

$$W_n(\cdot) \xrightarrow{d} W(\cdot) \text{ as } n \to \infty$$

where $W(\cdot)$ denotes a Wiener process defined on $[0,1]$.

Ito Processes

A process $X(t)$ is called an *Ito process* if it satisfies

$$\begin{aligned} dX(t) &= \mu(X(t),t)dt + \sigma(X(t),t)dW(t) \\ \mu(X(t),t) &= \text{drift function} \\ \sigma(X(t),t) &= \text{diffusion function} \end{aligned}$$

Such a process is also called a *stochastic differential equation* (SDE) or a *diffusion process* and has solution

$$X(t) = X(0) + \int_0^t \mu(X(t),t)dt + \int_0^t \sigma(X(t),t)dW(t)$$

The integral $\int_0^t \sigma(X(t),t)dW(t)$ is called a stochastic integral

Arithmetic Brownian Motion

This process was introduced by Bachelier (1900) and has the form

$$\begin{aligned} dX(t) &= \mu dt + \sigma dW(t) \\ X(t) &= X(0) + \mu \int_0^t dt + \sigma \int_0^t dW(t) = X(0) + \mu t + \sigma W(t) \end{aligned}$$

Note: This is not a good process for prices because it can become negative

Geometric Brownian Motion

To ensure positive values of $X(t)$, consider the following process for $\ln X(t)$

$$\begin{aligned} d\ln X(t) &= (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW(t) \\ \ln X(t) &= \ln X(0) + (\mu - \frac{1}{2}\sigma^2)t + \sigma W(t) \end{aligned}$$

Here, $X(t)$ given $X(s)$, $s < t$, is log-normally distributed. This process is used in the derivation of the Black-Scholes option pricing model.

Ornstein-Uhlenbeck (OU) Process

Arithmetic and geometric Brownian motion are non-stationary processes for $X(t)$. This is appropriate if $X(t)$ represents a price process. If $X(t)$ is to represent an interest rate process then $X(t)$ should be stationary. The simplest mean-reverting stationary process is the OU process

$$
\begin{aligned}
dX(t) &= \kappa(\alpha - X(t))dt + \sigma dW(t) \\
\mu(X(t),t) &= \kappa(\alpha - X(t)),\ \sigma(X(t),t) = \sigma \\
\kappa &= \text{mean reversion parameter} \\
\alpha &= \text{long-run mean}
\end{aligned}
$$

The OU process is a continuous-time version of an AR(1) process

Cox-Ingersoll-Ross (CIR) Square Root Process

A drawback of the OU process is that it can attain negative values. The CIR process avoids negative values by modifying the diffusion function of the OU process

$$
\begin{aligned}
dX(t) &= \kappa(\alpha - X(t))dt + \sigma\sqrt{X(t)}dW(t) \\
\mu(X(t), t) &= \kappa(\alpha - X(t)),\ \sigma(X(t), t) = \sigma\sqrt{X(t)} \\
\kappa &= \text{mean reversion parameter} \\
\alpha &= \text{long-run mean} \\
2\kappa\alpha &\geq \sigma^2 \text{ is required to avoid } X(t) \to 0
\end{aligned}
$$

Review of Non-Stochastic Differentiation

Let $G(x)$ be a differentiable function of the non-stochastic variables x and y. Using Taylor expansion about $(x + \Delta x, y + \Delta y)$, we have

$$\begin{aligned} \Delta G &= G(x + \Delta x, y + \Delta y) - G(x, y) \\ &= \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial y}\Delta y + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}(\Delta x)^2 \\ &\quad + \frac{1}{2}\frac{\partial^2 G}{\partial x \partial y}\Delta x \Delta y + \frac{1}{2}\frac{\partial^2 G}{\partial y^2}(\Delta y)^2 + \cdots \end{aligned}$$

Taking the limit as $\Delta x \to 0$ and $\Delta y \to 0$ gives to total derivative (differential)

$$dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy$$

Stochastic Differentiation: Ito's Lemma

Let $X(t)$ be an Ito process such that

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dW(t)$$

and let $G(X(t), t)$ be a differentiable function of $X(t)$ and t. The Taylor approximation to the differential is

$$\begin{aligned}\Delta G &= \frac{\partial G}{\partial X}\Delta X + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}(\Delta X)^2 \\ &\quad + \frac{1}{2}\frac{\partial^2 G}{\partial X \partial t}\Delta X \Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial t^2}(\Delta t)^2 + \cdots\end{aligned}$$

Consider the discretized version of $X(t)$

$$\Delta X = \mu \Delta t + \sigma \sqrt{\Delta t}\varepsilon, \ \varepsilon \sim N(0, 1)$$

Then

$$(\Delta X)^2 = \mu^2(\Delta t)^2 + \sigma^2 \Delta t \varepsilon^2 + 2\mu\sigma(\Delta t)^{3/2}\varepsilon = \sigma^2 \Delta t \varepsilon^2 + O((\Delta t)^{3/2})$$

Now,

$$\begin{aligned} E[\sigma^2 \Delta t \varepsilon^2] &= \sigma^2 \Delta t \\ var(\sigma^2 \Delta t \varepsilon^2) &= E[\sigma^4 (\Delta t)^2 \varepsilon^4] - E[\sigma^2 \Delta t \varepsilon^2]^2 = 2\sigma^4 (\Delta t)^2 \end{aligned}$$

so that

$$(\Delta X)^2 \rightarrow \sigma^2 dt \text{ as } \Delta t \rightarrow 0$$

Hence, keeping only terms involving ΔX and Δt and taking the limit as $\Delta t \rightarrow 0$ gives

$$dG = \frac{\partial G}{\partial X} dX + \frac{\partial G}{\partial t} dt + \frac{1}{2} \frac{\partial^2 G}{\partial X^2} \sigma^2 dt$$

Substituting in $dX = \mu dt + \sigma dW$ gives Ito's Lemma:

$$\begin{aligned} dG &= \frac{\partial G}{\partial X}(\mu dt + \sigma dW) + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}\sigma^2 dt \\ &= \left(\frac{\partial G}{\partial X}\mu + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}\sigma^2\right) dt + \frac{\partial G}{\partial X}\sigma dW \\ &= \left(\frac{\partial G}{\partial X}\mu(X(t),t) + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial X^2}\sigma^2(X(t),t)\right) dt \\ &\quad + \frac{\partial G}{\partial X}\sigma(X(t),t)dW(t) \end{aligned}$$

Example: Let $X(t) = W(t)$ and let $G(W(t)) = W(t)^2$. Then

$$\frac{\partial G}{\partial W} = 2W, \;\; \frac{\partial G}{\partial t} = 0, \;\; \frac{\partial^2 G}{\partial W^2} = 2$$

and from Ito's Lemma we have

$$\begin{aligned} dG &= dW(t)^2 = \left(2W \times 0 + 0 + \frac{1}{2} \times 2 \times 1\right) dt + 2W \times 1 \times dW \\ &= dt + 2W(t) dW(t) \end{aligned}$$

Example: Geometric Brownian Motion Again

Let $P(t)$ be the price of a stock at time t, and assume that it follows the process

$$\begin{aligned} dP(t) &= \mu(P(t),t)dt + \sigma(P(t),t)dW(t) \\ &= \mu P(t)dt + \sigma P(t)dW(t) \end{aligned}$$

where

$$\mu(P(t),t) = \mu P(t),\ \sigma(P(t),t) = \sigma P(t)$$

Note that this process for prices is an arithmetic Brownian motion for instantaneous returns

$$\frac{dP(t)}{P(t)} = \mu dt + \sigma dW(t)$$

Suppose we are interested in $G(P(t), t) = \ln P(t)$. By Ito's Lemma

$$\frac{\partial G}{\partial P} = \frac{1}{P}, \; \frac{\partial G}{\partial t} = 0, \frac{\partial^2 G}{\partial P^2} = -\frac{1}{P^2}$$

Then

$$\begin{aligned} d \ln P(t) &= \left(\frac{1}{P(t)} \mu P(t) - \frac{1}{2} \frac{1}{P(t)^2} \sigma^2 P(t)^2 \right) dt + \frac{1}{P(t)} \sigma P(t) dW(t) \\ &= \left(\mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dW(t) \end{aligned}$$

Note that

$$\begin{aligned} \ln P(t) &= \ln P(0) + \int_0^t \left(\mu - \frac{1}{2} \sigma^2 \right) dt + \sigma \int_0^t dW(t) \\ &= \ln P(0) + \left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W(t) \end{aligned}$$

Therefore, the change in log price (continuously compounded return) from time t to T is normally distributed

$$\ln P(T) - \ln P(t) \sim N\left(\left(\mu - \frac{1}{2}\sigma^2\right)(T-t), \sigma^2(T-t)\right)$$

Conditional on $\ln P(t)$, $\ln P(T)$ is normally distributed

$$\ln P(T) \sim N\left(\ln P(t) + \left(\mu - \frac{1}{2}\sigma^2\right)(T-t), \sigma^2(T-t)\right)$$

so that

$$\begin{aligned} E[P(T)|I_t] &= P_t \exp\left(\mu(T-t)\right) \\ var(P(T)|I_t) &= P_t^2 \exp\left(2\mu(T-t)\right)\left[\exp(\sigma^2(T-t)) - 1\right] \end{aligned}$$

Stochastic Integration

Like usual integration of a deterministic function, integration of a stochastic function is the opposite of differentiation. For example, let $W(t)$ be a Wiener process with increment $dW(t)$. Then

$$\int_0^t dW(s)ds = W(t) - W(0) = W(t)$$

Next, consider the stochastic integral

$$\int_0^t W(t)dW(t)$$

Because $W(t)$ is not differentiable

$$\int_0^t W(t)dW(t) \neq \frac{1}{2}W(t)^2$$

Recall, for $W(t)^2$ by Ito's Lemma we have

$$\begin{aligned} dW(t)^2 &= dt + 2W(t)dW(t) \Rightarrow \\ W(t)dW(t) &= \frac{1}{2}\Big(dW(t)^2 - dt\Big) \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^t W(t)dW(t) &= \frac{1}{2}\int_0^t dW(t)^2 - \frac{1}{2}\int_0^t dt \\ &= \frac{1}{2}\Big(W(t)^2 - t\Big) \end{aligned}$$