

Continuous-Time Derivative Pricing Models

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Outline

1. Derivative Pricing with Continuous-Time Models
2. Derivation of Black-Scholes (BS) SDE
3. BS Implied Volatility

Reading

- APDVP, chapters 13 and 14
- FMUNGD, chapters 10-12

Review

Continuous-time Ito process (stochastic differential equation, SDE)

$$\begin{aligned}dX(t) &= \mu(X(t), t)dt + \sigma(X(t), t)dW(t), \quad W(t) = \text{Wiener process} \\ &= \mu dt + \sigma dW(t)\end{aligned}$$

Ito's Lemma: For a continuous functional $G(X(t), t)$

$$dG(X(t), t) = \left(\frac{\partial G}{\partial x} \mu + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \sigma^2 \right) dt + \frac{\partial G}{\partial x} \sigma dW(t)$$

Application of Ito's Lemma to Derivative Pricing

Let $P(t)$ denote the continuous-time price process for an asset (e.g. stock) and assume that it follows a geometric Brownian motion

$$\begin{aligned}dP(t) &= \mu P(t)dt + \sigma P(t)dW(t) \Rightarrow \\r(t) &= \frac{dP(t)}{P(t)} = \mu dt + \sigma dW(t)\end{aligned}$$

Let $G(P(t), t)$ denote the price of a derivative security contingent on $P(t)$ (e.g. a call option on a stock).

Q: How to determine the no-arbitrage price of the derivative security?

A: Create a perfect hedge between derivative and underlying asset and use arbitrage arguments to value hedge portfolio

Creating Hedge Portfolio

Start by using Ito's Lemma

$$dG(P(t), t) = \left(\frac{\partial G}{\partial P} \mu P(t) + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial P^2} \sigma^2 P(t)^2 \right) dt + \frac{\partial G}{\partial P} \sigma P(t) dW(t)$$

and consider discrete approximations to continuous-time results

$$\begin{aligned} \Delta P_t &= \mu P_t \Delta t + \sigma P_t \sqrt{\Delta t} \varepsilon_t, \quad \varepsilon_t \sim N(0, 1) \\ \Delta G_t &= \left(\frac{\partial G_t}{\partial P_t} \mu P_t + \frac{\partial G_t}{\partial t} + \frac{1}{2} \frac{\partial^2 G_t}{\partial P_t^2} \sigma^2 P_t^2 \right) \Delta t + \frac{\partial G_t}{\partial P_t} \sigma P_t \sqrt{\Delta t} \varepsilon_t \end{aligned}$$

Note: both ΔP_t and ΔG_t depend on the same random component ε_t ! Hence, we can create a portfolio of P_t and G_t that eliminates the random component ε_t .

Hedge Portfolio

Short derivative and long $\frac{\partial G_t}{\partial P_t}$ shares of stock ($\frac{\partial G_t}{\partial P_t} =$ hedge “delta”). The current value of the hedge portfolio is

$$V_t = -G_t + \frac{\partial G_t}{\partial P_t} P_t$$

and the change in its value is

$$\Delta V_t = -\Delta G_t + \frac{\partial G_t}{\partial P_t} \Delta P_t$$

Substituting in the previous values for ΔG_t and ΔP_t gives

$$\begin{aligned}\Delta V_t &= - \left(\frac{\partial G_t}{\partial P_t} \mu P_t + \frac{\partial G_t}{\partial t} + \frac{1}{2} \frac{\partial^2 G_t}{\partial P_t^2} \sigma^2 P_t^2 \right) \Delta t - \frac{\partial G_t}{\partial P_t} \sigma P_t \sqrt{\Delta t} \varepsilon_t \\ &\quad + \frac{\partial G_t}{\partial P_t} (\mu P_t \Delta + \sigma P_t \sqrt{\Delta t} \varepsilon_t) \\ &= - \left(\frac{\partial G_t}{\partial t} + \frac{1}{2} \frac{\partial^2 G_t}{\partial P_t^2} \sigma^2 P_t^2 \right) \Delta t\end{aligned}$$

Note: ΔV_t does not depend on ε_t .

Result: Because $\Delta V_t = - \left(\frac{\partial G_t}{\partial t} + \frac{1}{2} \frac{\partial^2 G_t}{\partial P_t^2} \sigma^2 P_t^2 \right) \Delta t$ is deterministic (no random component), under no arbitrage conditions the hedge portfolio V_t must earn the risk-free rate of return over the time period Δt

$$\frac{\Delta V_t}{V_t} = r_f \Delta t \Rightarrow \Delta V_t = (r_f \Delta t) V_t$$

$r_f =$ risk free rate over Δt

Using

$$V_t = -G_t + \frac{\partial G_t}{\partial P_t} P_t$$

$$\Delta V_t = - \left(\frac{\partial G_t}{\partial t} + \frac{1}{2} \frac{\partial^2 G_t}{\partial P_t^2} \sigma^2 P_t^2 \right) \Delta t$$

gives

$$- \left(\frac{\partial G_t}{\partial t} + \frac{1}{2} \frac{\partial^2 G_t}{\partial P_t^2} \sigma^2 P_t^2 \right) \Delta t = r_f \Delta t \left(-G_t + \frac{\partial G_t}{\partial P_t} P_t \right)$$

Re-arranging gives the Black-Scholes partial differential equation (PDE) for derivative pricing:

$$\frac{\partial G_t}{\partial t} + r_f P_t \frac{\partial G_t}{\partial P_t} + \frac{1}{2} P_t^2 \frac{\partial^2 G_t}{\partial P_t^2} \sigma^2 = r_f G_t$$

The solution of the PDE depends on the boundary conditions of the derivative security.

1. European Call option. Right to purchase stock for exercise price K at maturity date T

$$G_T = \max(P_T - K, 0)$$

2. European Put option. Right to sell stock for exercise price K at maturity date T

$$G_T = \max(K - P_T, 0)$$

Solving the Black-Scholes PDE

The Black-Scholes PDE can be solved in a number of ways

1. Explicitly solve the PDE subject to boundary conditions. Applied Math types and Physicists are good at this (one type of Wall Street quant).
2. Numerically solve the PDE using approximation methods (e.g. finite difference methods)
3. Be tricky and use “Risk Neutral” pricing arguments (economists and finance people like this approach). This approach also lends itself to Monte Carlo solutions.

Risk Neutral Pricing

Recall, Black and Scholes assume that $P(t)$ follows a geometric Brownian motion so that the continuous-time rate of return is

$$\begin{aligned}r(t) &= \frac{dP(t)}{P(t)} = \mu dt + \sigma dW(t) \Rightarrow \\E[r(t)] &= \mu dt\end{aligned}$$

For a risky stock, the risk premium per unit time is

$$E[r(t)] - r_f dt = (\mu - r_f) dt$$

and, in general, depends on the risk preferences of market participants.

Black and Scholes made the following key observation: The Black-Scholes PDE

$$\frac{\partial G_t}{\partial t} + r_f P_t \frac{\partial G_t}{\partial P_t} + \frac{1}{2} P_t^2 \frac{\partial^2 G_t}{\partial P_t^2} \sigma^2 = r_f G_t$$

does not depend on the drift parameter μ from the geometric Brownian motion describing the evolution of $P(t)$. As a result, the solution to the PDE

- DOES NOT DEPEND ON THE RISK PREFERENCES OF MARKET PARTICIPANTS!!!!!!

This means that the derivative can be priced assuming market participants are risk neutral. This is the idea behind risk neutral pricing.

Implications of Risk Neutral Pricing

In a risk neutral world

1. The expected rate of return on all assets is the risk-free interest rate r_f
2. The present value of any future cash flow is obtained by discounting its expected value at the risk-free rate
3. Expected future cash flows are computed using risk-neutral probabilities (more generally, using the risk-neutral probability distribution associated with the cash flows)

Risk Neutral Pricing of European Call Option

Let $G(P(t), t)$ denote the price at t of a European call option, with exercise price K and maturity date T , on a stock with price $P(t)$ whose dynamic behavior is described by geometric Brownian motion. The expected value of the call option at maturity in a risk neutral world is

$$E_*[G(P(T), T)] = E_* [\max(P(T) - K, 0)]$$

$$E_*[\cdot] = \text{expectation under risk neutral probabilities}$$

$$E_*[G(P(T), T)] = \int_{-\infty}^{\infty} G(P(T), T) f_*(P(T)) dP(T)$$

Hence, the current value is present value of the expected payoff discounted at the risk-free rate

$$G(P(t), t) = e^{-r_f(T-t)} E_* [\max(P(T) - K, 0)]$$

Q: What is the risk neutral probability measure used to compute $E_*[\cdot]$?

A: Geometric Brownian motion of underlying stock with $\mu = r_f$

$$r(t) = \frac{dP(t)}{P(t)} = r_f dt + \sigma dW(t)$$

Recall, if $P(t)$ follows a geometric Brownian motion with drift r_f then by Ito's Lemma $\ln P(t)$ follows the process

$$d \ln P(t) = \left(r_f - \frac{\sigma^2}{2} \right) dt + \sigma dW(t)$$

and

$$\ln P(T) \sim N \left(\ln P(t) + \left(r_f - \frac{\sigma^2}{2} \right) (T - t), \sigma^2 (T - t) \right)$$

As a result, $P(T)$ given $P(t)$ is log-normally distributed

$$P(T)|P(t) \sim LN(\ln P(t) + \left(r_f - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t))$$

$$E_*[P(T)|P(t)] = P(t)e^{r_f(T-t)}$$

$$\text{var}_*(P(T)|P(t)) = P(t)^2 e^{2r_f(T-t)} \left\{ e^{\sigma^2(T-t)} - 1 \right\}$$

Therefore,

$$\begin{aligned} G(P(t), t) &= e^{-r_f(T-t)} E_* [\max(P(T) - K, 0)] \\ &= e^{-r_f(T-t)} \int_0^\infty \max(P(T) - K, 0) f_*(P(T)|P(t)) dP(T) \\ &= e^{-r_f(T-t)} \int_K^\infty (P(T) - K) f_*(P(T)|P(t)) dP(T) \end{aligned}$$

After a bunch of tedious calculus, it can be shown that

$$\begin{aligned}G(P(t), t) &= P(t)\Phi(h_+) - Ke^{r_f(T-t)}\Phi(h_-) \\h_+ &= \frac{\ln(P(t)/K) + (r_f + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\h_- &= h_+ - \sigma\sqrt{T-t} \\ \Phi(\cdot) &= \text{std normal CDF}\end{aligned}$$

Remarks

- $P(t)\Phi(h_+) =$ PV of receiving stock if $P(T) > K$; $Ke^{r_f(T-t)}\Phi(h_-) =$ PV of paying K for stock if $P(T) > K$
- Call option = hedge portfolio that is long $\Phi(h_+)$ units of stock and short $Ke^{r_f(T-t)}\Phi(h_-)$ units of risk-free bond

Implied Volatility

The Black-Scholes call option formula depends on 5 arguments

$$P(t), K, r_f, T - t, \text{ and } \sigma$$

At time t , all of the arguments are observable EXCEPT σ . For given values of $P(t), K, r_f, T - t$, define

$$BS_t(\sigma) = P(t)\Phi(h_+(\sigma)) - Ke^{r_f(T-t)}\Phi(h_-(\sigma))$$

This emphasizes that the call option price is fundamentally just a function of volatility. Given the observed market price of the call option at time t , c_t , we can extract unobserved volatility by solving:

$$BS_t(\sigma_{BS}) - c_t = 0$$

The value σ_{BS} is called the Black-Scholes implied volatility.

Remarks

1. σ_{BS} can be interpreted as the market's view on the underlying volatility of the stock
2. σ_{BS} is a forward looking view of volatility because it is based on the expected behavior of the stock from now until the option matures
3. $\sigma_{BS} = f(c_t, P(t), K, r_f, T - t)$.
4. If the Black-Scholes model is correct, then σ_{BS} should be the same for options with different strikes, K , and maturity dates, $T - t$