Amath 546/Econ 589
Analytical VaR for Bonds and Options

Eric Zivot

June 3, 2013

Reading

- FRF, chapter 6

Calculating Risk Measures for Bonds and Derivatives

- Up to now, we have only considered computing risk measures for primative assets like equities

- Many portfolios contain fixed income assets (e.g., government bonds, municipal bonds, corportate bonds, mortgage securities, etc.) and options or other derivative securities (e.g., futures, fowards, swaps, etc.)

- Calculating risk measures for fixed income assets and derivatives is more involved than calculating risk measures for equities

- Here, we focus on calculating risk measures for default-free bonds and call and put options

Calculating Risk Measures for Bonds and Options

- Bonds and options have values that change with the passage of time
  - PV of bond's cash flows change with time even if interest rates stay constant
  - BS option pricing formula depends on the option's time to maturity
- Have to be careful when computing volatility for bonds and options
  - Analytic approximations
  - Simulation methods

Calculating VaR for Bonds

Consider a default-free coupon bond maturing in T years with cash flows $c_1, \ldots, c_T$. Assuming a flat yield curve with annual interest rate r, the bond price is the present value of the future cash flows

$$P(r, T) = \frac{c_1}{(1+r)} + \frac{c_2}{(1+r)^2} + \cdots + \frac{c_2}{(1+r)^T} = \sum_{t=1}^{T} \frac{c_t}{(1+r)^t}$$

Bond prices change when:

- interest rates, r, change (main risk)
- time changes (although not for all coupon bonds) (secondary risk)

Result: For short-time periods, the risk of holding bonds is interest rate risk.

Problem: Symmetric interest rate changes (e.g., 1% increase or 1% decrease) cause asymmetric changes in $P(r, T)$

Example: Bond with face value \$1000, T=50 and annual coupon of \$30

r	$P(r,T)$	$\lvert\Delta P\rvert$
1%	892	408
3%	484	
5%	319	165

- r dropping from 3% to 1% $\Rightarrow |\Delta P| = 408$
- r rising from 3% to 5% $\Rightarrow |\Delta P| = 165$

Approximating the Price Impact of Interest Rates

Let $P(r)$ denote the price of a default free bond as a function of r. For a small interest rate change $dr \approx 0$, a 1st order TSE of $P(r+dr)$ gives

$$\begin{aligned} P(r+dr) &\approx P(r) + P'(r)dr \\ P'(r) &= \frac{d}{dr}P(r) \end{aligned}$$

Hence,

$$\%\Delta P(r) \approx \frac{P(r+dr) - P(r)}{P(r)} = \frac{P'(r)}{P(r)}dr$$

Example: Pure discount bond with face value \$1 maturing at time T

$$\begin{aligned} P(r,T) &= \$1(1+r)^{-T} \\ P'(r) &= \frac{dP(r)}{dr} = -T(1+r)^{-T-1} = -T(1+r)^{-T}(1+r)^{-1} \\ &= -T(1+r)^{-1}P(r,T) \\ &\Rightarrow \%\Delta P(r) \approx \frac{P'(r)}{P(r,T)}dr = -T(1+r)^{-1}dr \end{aligned}$$

Here,

$$\begin{aligned} T &= \text{duration of bond} = D \\ T(1+r)^{-1} &= \text{modified duration of bond} \\ &= (1+r)^{-1} \times D = D^* \end{aligned}$$

Then

$$\%\Delta P(r) \approx \frac{P'(r)}{P(r,T)} = -D^* dr$$

Approximating the Price Impact of Interest Rates

Let $P(r)$ denote the price as a function of r for a general default-free bond.

Define *Modified Duration as*

$$D^* = \frac{-P'(r)}{P(r)}$$

For a small change in interest rates dr, we approximate the bond return as

$$\%\Delta P(r) \approx \frac{P'(r)}{P(r,T)} = -D^* dr$$

Daily VaR of Bond Investment

Let $dr = r_t - r_{t-1} =$ daily change in the risk-free annual interest rate

Assume

$$dr = r_t - r_{t-1} \sim N(0, \sigma_r^2)$$

Then for a bond with price $P(r)$ we have the approximation

$$R^{bond} = \%\Delta P(r) \approx -D^* \times dr \sim N(0, (D^* \sigma_r)^2)$$

and so the daily VaR at the $\alpha \times 100\%$ confidence level is

$$VaR_{\alpha}^{bond} = (D^* \sigma_r) \times q_{1-\alpha}^{Z} \times P(r)$$

Accuracy of Duration-Based VaR

- Depends on magnitude of D^*.
  - For small D^* (short-maturity bonds), the approximation is very good
  - For large D^* (long-maturity bonds), the approximation can be poor
- Depends on σ_r
  - For small σ_r, the approximation is very good
  - For large σ_r, the approximation can be poor

Calculating VaR for Options

Consider call and put options on a stock

Definition 1 *A European Call option gives the holder the right but not the obligation to buy the stock at the strike price* X *at the expiration date* T

Definition 2 *A European Put option gives the holder the right but not the obligation to sell the stock at the strike price* X *at the expiration date* T

Black-Scholes Option Pricing Formula

European call and put options can be priced using the BS formula

$$
\begin{aligned}
Call_t &= P_t \Phi(d_1) - X e^{-r(T-t)} \Phi(d_2) \\
Put_t &= X e^{-r(T-t)} - P_t + Call_t \\
P_t &= \text{stock price at t} \\
X &= \text{ exercise price} \\
r &= \text{annual risk free rate} \\
\sigma_a &= \text{annual stock return vol}
\end{aligned}
$$

where

$$
d_1 = \frac{\ln(P_t/X) + (r + \sigma_a^2/2)(T-t)}{\sigma_a \sqrt{T-t}}, \; d_2 = d_1 - \sigma_a \sqrt{T-t}
$$

Delta

Let $g(P)$ denote the $Call_t$ or Put_t as a function of P.

The first order sensitivity of an option w.r.t. the underlying price is called delta:

$$\begin{aligned} \Delta &= \frac{\partial g(P)}{\partial P} = \begin{cases} \Phi(d_1) > 0 & \text{call} \\ \Phi(d_1) - 1 < 0 & \text{put} \end{cases} \\ &\Rightarrow \partial g(P) \approx g(P + dP) - g(P) \approx \Delta dP \end{aligned}$$

Remarks:

- Deep in-the-money options: $\Delta \approx 1$ for calls, $\Delta \approx -1$ for puts
- At-the-money options: $\Delta \approx 0.5$ for calls, $\Delta \approx -0.5$ for puts
- Deep out-of-the money options: $\Delta \approx 0$ for calls and puts
- Approximation is accurate for prices close to the price at which Δ is calculated
- Approximation is more accurate for longer maturity options and when the option is deep in or out of the money

Delta-Normal VaR for Options

Let $dP = P_t - P_{t-1}$ = daily change in stock price, and let $g(P)$ denote the $Call_t$ or Put_t as a function of P. Then

$$dg(P) = g_t - g_{t-1} \approx \Delta \times dP = \Delta(P_t - P_{t-1})$$

Assume

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} \sim N(0, \sigma_d^2)$$

To determine the option VaR at the $\alpha \times 100\%$ confidence level we solve

$$\begin{aligned}
1-\alpha &= \Pr(g_t - g_t \leq -VaR_\alpha^{option}) \\
&= \Pr(\Delta(P_t - P_{t-1}) \leq -VaR_\alpha^{option}) \\
&= \Pr(\Delta P_{t-1} R_t \leq -VaR_\alpha^{option}) \\
&= \Pr\left(\frac{R_t}{\sigma_d} \leq -\frac{1VaR_\alpha^{option}}{\Delta P_{t-1}\sigma_d}\right) \\
&= \Pr\left(Z \leq -\frac{1VaR_\alpha^{option}}{\Delta P_{t-1}\sigma_d}\right)
\end{aligned}$$

Hence

$$q_{1-\alpha}^{Z} = -\frac{1VaR_{\alpha}^{option}}{\Delta P_{t-1}\sigma_d}$$

Then the daily option VaR for holding one unit of the stock at the $\alpha \times 100\%$ confidence level is

$$\begin{aligned} VaR_{\alpha}^{option} &= -|\Delta| \times \sigma_d \times q_{1-\alpha}^{Z} \times P_{t-1} \\ &= -|\Delta| \times VaR_{\alpha}^{stock} \end{aligned}$$