Amath 546/Econ 589
Backtesting Risk Models

Eric Zivot

April 29, 2013

Lecture Outline

- Backtesting terminology

- Backtesting VaR

- Backtesting ES

Note: A nice review of VaR backtesting is Campbell, S. (2005). "A Review of Backtesting and Backtesting Procedures", Federal Reserve Board (thanks to Ian Kaplan for reference).

Regulatory Framework

- The current regulatory framework requires that financial institutions use their own internal risk models to calculate and report their 99% value-at-risk, $\text{VaR}_{.99}$, over a 10 day horizon.

- Market risk capital requirements are directly linked to both the estimated level of portfolio risk as well as the VaR model's performance on backtests.

- Specifcally, the risk based capital requirement is set as the larger of either the bank's current assessment of the 99% VaR over the next 10 trading days or a multiple of the bank's average reported 99% VaR over the previous 60 trading days plus an additional amount that reflects the underlying credit risk of the bank's portfolio.

Market risk capital

$$MRC_t = \mathsf{max}\left(VaR_{.99,t},\ S_t \times \frac{1}{60}\sum_{i=0}^{59} VaR_{.99,t-i}\right) + c$$

where S_t = multiplication factor that depends on VaR backtesting results. Specifically, let N = number of 99% VaR violations in the previous 250 trading days. Then

$$S_t = \begin{cases} 3 & \text{if } N \leq 4 \text{ (green)} \\ 3 + 0.2(N-4) & \text{if } 5 \leq N \leq 9 \text{ (yellow)} \\ 4 & \text{if } 10 < N \text{ (red)} \end{cases}$$

Result: If the VaR model underforecasts risks, then S_t compensates for this in setting regulatory capital.

Backtesting Terminology

Q: How does a VaR model's forecasts perform over an historical period?

A: Compare ex ante VaR forecast over rolling windows to ex post realized return

Let $t = 1, \ldots, T$ denote the sample size.

Definition 1 *(Estimation window)* $[1, \ldots, W_E]$ = *observations used to initially estimate risk model.* W_E = *number of observations in estimation window.*

Definition 2 *(Testing window)* $[W_E + 1, \ldots, T]$ = *observations over which risk is forecast.* W_T = *number of observations in testing window.*

Note: $W_E + W_T = T$

Backtesting VaR Models

- Define the VaR violation ("Hit") indicator

$$
\begin{aligned}
H_t &= 1(r_t < VaR_{\alpha,t}) = \begin{cases} 1 & r_t < VaR_{\alpha,t} \\ 0 & r_t \geq VaR_{\alpha,t} \end{cases} \\
VaR_{\alpha,t} &= q^r_{1-\alpha,t} = q^r_{p,t}, \ p = 1 - \alpha
\end{aligned}
$$

- VaR forecasts are efficient wrt I_t if

$$
\begin{aligned}
E[H_t|I_{t-1}] &= \Pr(H_t = 1|I_{t-1}) = 1 - \alpha = p \\
&\Rightarrow H_t|I_{t-1} \sim Bernoulli(p), t = W_E + 1, \ldots, T
\end{aligned}
$$

- n_1 = number of sample VaR violations, $n_0 = W_T - n_1$. Note: $E[n_1] = p \times W_T$ and $\hat{p}_{mle} = n_1/W_T$

Sample Estimates

$$
\begin{aligned}
n_1 &= \text{number of VaR violations} \\
\hat{p}_{mle} &= n_1/W_T = \text{fraction of sample with } H_t = 1 \\
VR &= \frac{n_1}{p \times W_T} = \frac{\text{Observed number of violations}}{\text{Expected number of violations}}
\end{aligned}
$$

Note:

$$
\begin{aligned}
VR &= 1: \text{ VaR model correctly forecasts risk} \\
VR &< 1: \text{ VaR model overforecasts risk} \\
VR &> 1: \text{ VaR model underforecasts risk}
\end{aligned}
$$

Test of Unconditional Coverage (Kupiec Test)

- Hypothesis to be tested

$$H_0 : E[H_t] = p \text{ vs. } H_1 : E[H_t] \neq p$$

- Bernoulli likelihood

$$f(p|H_{W_E+1}, \ldots, H_T) = p^{n_1}(1-p)^{W_T - n_1}$$

- LR test for unconditional coverage

$$LR_{uc} = 2\left[\ln f(\hat{p}_{mle}|H_{W_E+1}, \ldots, H_T) - \ln f(p|H_{W_E+1}, \ldots, H_T)\right]$$

Under H_0, $LR_{uc} \sim \chi^2(1)$. Reject $H_0 : E[H_t] = p$ at 5% level if $LR_{uc} > \chi^2_{.95}(1) = 3.84$

Test of Independence

- VaR forecasts that do not take temporal volatility dependence into account may be correct on average, but will produce violation clusters

- A test of independence is a test of no violation clusters (no dependence in VaR violations)

- Christoffersen (1998) models H_t as a binary first order Markov chain with transition matrix

$$\Pi = \begin{bmatrix} 1-\pi_{01} & \pi_{01} \\ 1-\pi_{11} & \pi_{11} \end{bmatrix}, \ \pi_{ij} = \Pr(H_t = j | H_{t-1} = i)$$

- Approximate joint likelihood conditional on first observation is

$$
\begin{aligned}
L(\Pi | H_{W_E+2}, \ldots, H_T) &= (1-\pi_{01})^{n_{00}} \pi_{01}^{n_{01}} (1-\pi_{11})^{n_{10}} \pi_{11}^{n_{11}} \\
n_{ij} &= \sum_{t=W_E+2}^{T} 1(H_t = i | H_{t-1} = j)
\end{aligned}
$$

- MLEs of transition probabilities

$$
\begin{aligned}
\hat{\pi}_{01,mle} &= \frac{n_{01}}{n_{00}+n_{01}} = \%\text{ violations immediately following no violation} \\
\hat{\pi}_{11,mle} &= \frac{n_{11}}{n_{10}+n_{11}} = \%\text{ violations immediately following a violation}
\end{aligned}
$$

- Under null of independence, $\pi_{01} = \pi_{11} \equiv \pi_0$ (% violations immediately following no violation = % violations immediately following a violation)

$$\begin{aligned} L(\pi_0|H_{W_E+2},\ldots,H_T) &= (1-\pi_{01})^{(n_{00}+n_{10})}\pi_{01}^{n_{01}+n_{11}} \\ \hat{\pi}_0 &= \hat{p}_{mle} = n_1/W_T \end{aligned}$$

- LR test for independence of VaR violations is

$$LR_{ind} = 2\left[\ln L(\hat{\Pi}_{mle}|H_{W_E+2},\ldots,H_T) - \ln L(\hat{\pi}_0|H_{W_E+2},\ldots,H_T)\right]$$

Under $H_0 : LR_{ind} \sim \chi^2(1)$. Reject $H_0 : \pi_{01} = \pi_{11} \equiv \pi_0$ if $LR_{ind} > \chi^2_{.95}(1) = 3.84$

Joint Test of Conditional Coverage and Independence

- Because $\hat{\pi}_0$ is unconstrained, the LR test for independence does not take correct conditional coverage into account.

- To jointly test correct conditional coverage $E[H_t|I_{t-1}] = \alpha$ along with independence, Christoffersen suggests using

$$\begin{aligned} LR_{cc} &= 2\left[\ln L(\hat{\pi}_0|H_{W_E+2},\ldots,H_T) - \ln f(p|H_{W_E+2},\ldots,H_T)\right] \\ &= LR_{uc} + LR_{ind} \sim \chi^2(2) \end{aligned}$$

Backtesting ES

Problem: Harder to backtest ES than VaR because ES is an expectation rather than a single quantile

Method to Backtest Shortfall

Consider the normalized shortfall when $r_t \leq VaR_\alpha$

$$NS_t = \frac{r_t}{ES_{\alpha,t}}$$

From the definition of ES, we have

$$\frac{E[r_t | r_t < VaR_\alpha]}{ES_{\alpha,t}} = 1$$

Hence, in a correctly specified model we should have

$$E[NS_t] = 1$$