Amath 546/Econ 589
Backtesting Risk Models

Eric Zivot

April 29, 2013
Lecture Outline

• Backtesting terminology

• Backtesting VaR

• Backtesting ES

Regulatory Framework

- The current regulatory framework requires that financial institutions use their own internal risk models to calculate and report their 99% value-at-risk, \( \text{VaR}_{.99} \), over a 10 day horizon.

- Market risk capital requirements are directly linked to both the estimated level of portfolio risk as well as the VaR model’s performance on backtests.

- Specifically, the risk based capital requirement is set as the larger of either the bank’s current assessment of the 99% VaR over the next 10 trading days or a multiple of the bank’s average reported 99% VaR over the previous 60 trading days plus an additional amount that reflects the underlying credit risk of the bank’s portfolio.
Market risk capital

\[ MRC_t = \max \left( VaR_{.99,t}, S_t \times \frac{1}{60} \sum_{i=0}^{59} VaR_{.99,t-i} \right) + c \]

where \( S_t = \) multiplication factor that depends on VaR backtesting results. Specifically, let \( N = \) number of 99% VaR violations in the previous 250 trading days. Then

\[
S_t = \begin{cases} 
3 & \text{if } N \leq 4 \text{ (green)} \\
3 + 0.2(N - 4) & \text{if } 5 \leq N \leq 9 \text{ (yellow)} \\
4 & \text{if } 10 < N \text{ (red)} 
\end{cases}
\]

Result: If the VaR model underforecasts risks, then \( S_t \) compensates for this in setting regulatory capital.
Backtesting Terminology

Q: How does a VaR model’s forecasts perform over an historical period?

A: Compare ex ante VaR forecast over rolling windows to ex post realized return

Let $t = 1, \ldots, T$ denote the sample size.

Definition 1 (Estimation window) $[1, \ldots, W_E] =$ observations used to initially estimate risk model. $W_E =$ number of observations in estimation window.

Definition 2 (Testing window) $[W_E + 1, \ldots, T] =$ observations over which risk is forecast. $W_T =$ number of observations in testing window.

Note: $W_E + W_T = T$
Backtesting VaR Models

- Define the VaR violation (“Hit”) indicator

\[
H_t = 1(r_t < \text{VaR}_{\alpha,t}) = \begin{cases} 
1 & r_t < \text{VaR}_{\alpha,t} \\
0 & r_t \geq \text{VaR}_{\alpha,t} 
\end{cases}
\]

\[
\text{VaR}_{\alpha,t} = q_{1-\alpha,t} = q_{p,t}, \quad p = 1 - \alpha
\]

- VaR forecasts are efficient wrt \( I_t \) if

\[
E[H_t | I_{t-1}] = \Pr(H_t = 1 | I_{t-1}) = 1 - \alpha = p
\]

\[
\Rightarrow H_t | I_{t-1} \sim \text{Bernoulli}(p), \quad t = W_E + 1, \ldots, T
\]

- \( n_1 \) = number of sample VaR violations, \( n_0 = W_T - n_1 \). Note: \( E[n_1] = p \times W_T \) and \( \hat{p}_{mle} = n_1 / W_T \)
Sample Estimates

\[ n_1 = \text{number of VaR violations} \]
\[ \hat{p}_{mle} = \frac{n_1}{W_T} = \text{fraction of sample with } H_t = 1 \]
\[ VR = \frac{n_1}{p \times W_T} = \text{Observed number of violations} \]
\[ VR = \frac{1}{p \times W_T} = \text{Expected number of violations} \]

Note:

\[ VR = 1 : \text{VaR model correctly forecasts risk} \]
\[ VR < 1 : \text{VaR model overforecasts risk} \]
\[ VR > 1 : \text{VaR model underforecasts risk} \]
Test of Unconditional Coverage (Kupiec Test)

- Hypothesis to be tested
  \[ H_0 : E[H_t] = p \text{ vs. } H_1 : E[H_t] \neq p \]

- Bernoulli likelihood
  \[ f(p|H_{W_E+1}, \ldots, H_T) = p^{n_1}(1 - p)^{W_T-n_1} \]

- LR test for unconditional coverage
  \[ LR_{uc} = 2 \left[ \ln f(\hat{p}_{mle}|H_{W_E+1}, \ldots, H_T) - \ln f(p|H_{W_E+1}, \ldots, H_T) \right] \]

  Under \( H_0 \), \( LR_{uc} \sim \chi^2(1) \). Reject \( H_0 : E[H_t] = p \) at 5% level if \( LR_{uc} > \chi_{.95}^2(1) = 3.84 \)
Test of Independence

- VaR forecasts that do not take temporal volatility dependence into account may be correct on average, but will produce violation clusters.

- A test of independence is a test of no violation clusters (no dependence in VaR violations).

- Christoffersen (1998) models $H_t$ as a binary first order Markov chain with transition matrix

\[
\Pi = \begin{bmatrix}
1 - \pi_{01} & \pi_{01} \\
1 - \pi_{11} & \pi_{11}
\end{bmatrix}, \quad \pi_{ij} = \Pr(H_t = j | H_{t-1} = i)
\]
• Approximate joint likelihood conditional on first observation is

\[
L(\Pi|H_{W_E+2}, \ldots, H_T) = (1 - \pi_{01})^{n_{00}}\pi_{01}^{n_{01}}(1 - \pi_{11})^{n_{10}}\pi_{11}^{n_{11}}
\]

\[
n_{ij} = \sum_{t=W_E+2}^{T} 1(H_t = i|H_{t-1} = j)
\]

• MLEs of transition probabilities

\[
\hat{\pi}_{01,mle} = \frac{n_{01}}{n_{00} + n_{01}} = \% \text{ violations immediately following no violation}
\]

\[
\hat{\pi}_{11,mle} = \frac{n_{11}}{n_{10} + n_{11}} = \% \text{ violations immediately following a violation}
\]
- Under null of independence, \( \pi_{01} = \pi_{11} \equiv \pi_0 \) (% violations immediately following no violation = % violations immediately following a violation)

\[
L(\pi_0 | H_W, H_T) = (1 - \pi_{01})(n_{00} + n_{10})^{n_{01} + n_{11}}
\]
\[
\hat{\pi}_0 = \hat{p}_{mle} = n_1 / W_T
\]

- LR test for independence of VaR violations is

\[
LR_{ind} = 2 \left[ \ln L(\hat{\pi}_{mle} | H_W, H_T) - \ln L(\hat{\pi}_0 | H_W, H_T) \right]
\]

Under \( H_0 : LR_{ind} \sim \chi^2(1) \). Reject \( H_0 : \pi_{01} = \pi_{11} \equiv \pi_0 \) if \( LR_{ind} > \chi^2_{0.95}(1) = 3.84 \)
Joint Test of Conditional Coverage and Independence

- Because $\hat{\pi}_0$ is unconstrained, the LR test for independence does not take correct conditional coverage into account.

- To jointly test correct conditional coverage $E[H_t|I_{t-1}] = \alpha$ along with independence, Christoffersen suggests using

$$LR_{cc} = 2 \left[ \ln L(\hat{\pi}_0|H_{WE+2}, \ldots, H_T) - \ln f(p|H_{WE+2}, \ldots, H_T) \right]$$

$$= LR_{uc} + LR_{ind} \sim \chi^2(2)$$
**Backtesting ES**

Problem: Harder to backtest ES than VaR because ES is an expectation rather than a single quantile

Method to Backtest Shortfall

Consider the normalized shortfall when \( r_t \leq VaR_\alpha \)

\[
NS_t = \frac{r_t}{ES_{\alpha,t}}
\]

From the definition of ES, we have

\[
\frac{E[r_t | r_t < VaR_\alpha]}{ES_{\alpha,t}} = 1
\]

Hence, in a correctly specified model we should have

\[
E[NS_t] = 1
\]