

1 Choosing the Lag Length for the ADF Test

An important practical issue for the implementation of the ADF test is the specification of the lag length p .

- If p is too small then the remaining serial correlation in the errors will bias the test.
- If p is too large then the power of the test will suffer.
- Monte Carlo experiments suggest it is better to error on the side of including too many lags.

Ng and Perron “Unit Root Tests in ARMA Models with Data-Dependent Methods for the Selection of the Truncation Lag,” JASA, 1995.

- Set an upper bound p_{\max} for p .
- Estimate the ADF test regression with $p = p_{\max}$.
- If the absolute value of the t-statistic for testing the significance of the last lagged difference is greater than 1.6 then set $p = p_{\max}$ and perform the unit root test. Otherwise, reduce the lag length by one and repeat the process.
- A common rule of thumb for determining p_{\max} , suggested by Schwert (1989), is

$$p_{\max} = \left[12 \cdot \left(\frac{T}{100} \right)^{1/4} \right]$$

where $[x]$ denotes the integer part of x . However, this choice is *ad hoc*!

Ng and Perron “Lag Length Selection and the Construction of Unit Root Tests with Good Size and Power,” ECTA, 2001.

- Select p as $p_{mic} = \arg \min_{p \leq p_{\max}} MAIC(p)$ where

$$MAIC(p) = \ln(\hat{\sigma}_p^2) + \frac{2(\tau_T(p) + p)}{T - p_{\max}}$$

$$\tau_T(p) = \frac{\hat{\pi}^2 \sum_{t=p_{\max}+1}^T y_{t-1}}{\hat{\sigma}_p^2}$$

$$\hat{\sigma}_p^2 = \frac{1}{T - p_{\max}} \sum_{t=p_{\max}+1}^T \hat{\varepsilon}_t^2$$

- Procedure is implemented in Eviews and S+FinMetrics 2.0

2 Phillips-Perron Unit Root Tests

The test regression for the PP tests is

$$\begin{aligned}\Delta y_t &= \beta' \mathbf{D}_t + \pi y_{t-1} + u_t \\ u_t &\sim I(0)\end{aligned}$$

The PP tests correct for any serial correlation and heteroskedasticity in the errors u_t of the test regression by directly modifying the test statistics $t_{\pi=0}$ and $T\hat{\pi}$. These modified statistics, denoted Z_t and Z_π , are given by

$$\begin{aligned}Z_t &= \left(\frac{\hat{\sigma}^2}{\hat{\lambda}^2}\right)^{1/2} \cdot t_{\pi=0} - \frac{1}{2} \left(\frac{\hat{\lambda}^2 - \hat{\sigma}^2}{\hat{\lambda}^2}\right) \cdot \left(\frac{T \cdot SE(\hat{\pi})}{\hat{\sigma}^2}\right) \\ Z_\pi &= T\hat{\pi} - \frac{1T^2 \cdot SE(\hat{\pi})}{2\hat{\sigma}^2} (\hat{\lambda}^2 - \hat{\sigma}^2)\end{aligned}$$

The terms $\hat{\sigma}^2$ and $\hat{\lambda}^2$ are consistent estimates of the variance parameters

$$\begin{aligned}\sigma^2 &= \lim_{T \rightarrow \infty} T^{-1} \sum_{t=1}^T E[u_t^2] \\ \lambda^2 &= \lim_{T \rightarrow \infty} \sum_{t=1}^T E \left[T^{-1} S_T^2 \right] = lrv \\ S_T &= \sum_{t=1}^T u_t\end{aligned}$$

The sample variance of the least squares residual \hat{u}_t is a consistent estimate of σ^2 , and the Newey-West long-run variance estimate of u_t using \hat{u}_t is a consistent estimate of λ^2 .

Result: Under the null hypothesis that $\pi = 0$, the PP Z_t and Z_π statistics have the same asymptotic distributions as the ADF t-statistic and normalized bias statistics.

3 Some Problems with Unit Root Tests

The ADF and PP tests are asymptotically equivalent but may differ substantially in finite samples due to the different ways in which they correct for serial correlation in the test regression.

- Schwert “Test for Unit Roots: A Monte Carlo Investigation,” *JBES*, 1989, finds that if $\Delta y_t \sim \text{ARMA}$ with a large and negative MA component, then the ADF and PP tests are severely size distorted (reject $I(1)$ null much too often when it is true) and that the PP tests are more size distorted than the ADF tests.
- Perron and Ng “Useful Modifications to Some Unit Root Tests with Dependent Errors and their Local Asymptotic Properties,” *RESTUD*, (1996), suggest useful modifications to the PP tests to mitigate size distortion of PP tests.

- ADF and PP tests have very low power against $I(0)$ alternatives that are close to being $I(1)$.
- The power of unit root tests diminish as deterministic terms are added to the test regressions.
- For maximum power against very persistent alternatives the so-called efficient unit root tests should be used.
 - Elliot, Rothenberg, and Stock “Efficient Tests for an Autoregressive Unit Root,” *ECTA*, 1996.
 - Ng and Perron “Lag Length Selection and the Construction of Unit Root Tests with Good Size and Power,” *ECTA*, 2001.
 - Tests are implemented in Eviews and S+FinMetrics 2.0.

4 Stationarity Tests

Kwiatkowski, Phillips, Schmidt and Shin (1992) derive their test by starting with the model

$$\begin{aligned}y_t &= \beta' \mathbf{D}_t + \mu_t + u_t, \quad u_t \sim I(0) \\ \mu_t &= \mu_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim WN(0, \sigma_\varepsilon^2) \\ \mathbf{D}_t &= \text{deterministic components}\end{aligned}$$

The hypotheses to be tested are

$$\begin{aligned}H_0 &: \sigma_\varepsilon^2 = 0 \Rightarrow y_t \sim I(0) \\ H_1 &: \sigma_\varepsilon^2 > 0 \Rightarrow y_t \sim I(1)\end{aligned}$$

The KPSS test statistic is the Lagrange multiplier (LM) or score statistic for testing $\sigma_\varepsilon^2 = 0$:

$$KPSS = \left(T^{-2} \sum_{t=1}^T \hat{S}_t^2 \right) / \hat{\lambda}^2$$
$$\hat{S}_t = \sum_{j=1}^t \hat{u}_j$$

where

- \hat{u}_t is the residual of a regression of y_t on \mathbf{D}_t
- $\hat{\lambda}^2$ is a consistent estimate of the long-run variance of u_t using \hat{u}_t .

Asymptotic results: Assume $H_0: y_t \sim I(0)$ is true.

- If $\mathbf{D}_t = 1$ then

$$KPSS \Rightarrow \int_0^1 V_1(r) dr$$
$$V_1(r) = W(r) - rW(1)$$

- If $\mathbf{D}_t = (1, t)'$ then

$$KPSS \xrightarrow{d} \int_0^1 V_2(r) dr$$
$$V_2(r) = W(r) + r(2 - 3r)W(1)$$
$$+ 6r(r^2 - 1) \int_0^1 W(s) ds$$

Note: $V_1(r) = W(r) - rW(1)$ is called a standard Brownian bridge. It satisfies $V_1(0) = V_1(1) = 0$.

Distribution	Right tail quantiles				
	0.90	0.925	0.950	0.975	0.99
$\int_0^1 V_1(r)dr$	0.349	0.396	0.446	0.592	0.762
$\int_0^1 V_2(r)dr$	0.120	0.133	0.149	0.184	0.229

Table 1: Quantiles of the distribution of the KPSS statistic

- Critical values from the asymptotic distributions must be obtained by simulation methods
- The stationary test is a one-sided right-tailed test so that one rejects the null of stationarity at the $100 \cdot \alpha\%$ level if the KPSS test statistic is greater than the $100 \cdot (1 - \alpha)\%$ quantile from the appropriate asymptotic distribution.