Econ 583
Multiple Equation GMM Examples

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Example: Simultaneous Equations Model (Klein's Model 1)

Reference: Berndt (1991, Chapter 10), *The Practice of Econometrics*, Addison Wesley.

Three equation macro model with five accounting identies (annual data):

1. consumption function

2. investment function

3. labor demand function

Consumption function

$$\begin{aligned} CN_i &= \alpha_0 + \alpha_1(W_{i1} + W_{i2}) + \alpha_2 P_i + \alpha_3 P_{i-1} + \varepsilon_{i1} \\ W_{i1} + W_{i2} &= \text{private + public wage income} \\ P_i &= \text{profit income} \end{aligned}$$

Net Investment function

$$\begin{aligned} I_i &= \beta_0 + \beta_1 P_i + \beta_2 P_{i-1} + \beta_3 K_{i-1} + \varepsilon_{i2} \\ K_i &= \text{capital stock} \end{aligned}$$

Labor Demand (wage bill)

$$\begin{aligned} W_{i1} &= \gamma_0 + \gamma_1 E_i + \gamma_2 E_{i-1} + \gamma_3(i - 1931) + \varepsilon_{i3} \\ E_i &= \text{private product} \end{aligned}$$

Accounting Identies

$$\begin{aligned} \text{Total product} &: \quad Y_i + TX_i = CN_i + I_i + G_i \\ \text{Income} &: \quad Y_i = P_i + W_i \\ \text{Capital} &: \quad K_i = I_i + K_{i-1} \\ \text{Wages} &: \quad W_i = W_{i1} + W_{i2} \\ \text{Private product} &: \quad E_i = Y_i + TX_i - W_{i2} \end{aligned}$$

where

$TX_i =$ indirect business taxes

$G =$ net government demand.

Remarks:

1. W_{i1} and P_i are endogenous variables in consumption function

2. P_i is endogenous in investment function

3. E_i is endogenous in labor demand function

4. $\varepsilon_{1i}, \varepsilon_{2i}$ and ε_{3i} may be correlated and heteroskedastic

Total endogenous variables

$$CN_i, I_i, W_{i1}, Y_i, P_i, K_i, W_i, E_i$$

Total predetermined variables (potential instruments)

$$1, G_i, W_{2i}, TX_i, i - 1931, K_{i-1}, P_{i-1}, E_{i-1}$$

3SLS Specification in Hayashi Notation

Consumption function

$$\begin{aligned} y_{i1} &= CN_i \\ \mathbf{z}_{i1} &= (1, W_i, P_i, P_{i-1})', \ \boldsymbol{\delta}_1 = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)', \ L_1 = 4 \end{aligned}$$

Investment function

$$\begin{aligned} y_{i2} &= I_i \\ \mathbf{z}_{i2} &= (1, W_i, P_i, P_{i-1})', \ \boldsymbol{\delta}_2 = (\beta_0, \beta_1, \beta_2, \beta_3)', \ L_2 = 4 \end{aligned}$$

Labor demand function

$$\begin{aligned} y_{i3} &= W_{1i} \\ \mathbf{z}_{i3} &= (1, E_i, E_{i-1}, i - 1931)', \ \boldsymbol{\delta}_3 = (\gamma_0, \gamma_1, \gamma_2, \gamma_3)', \ L_3 = 4 \end{aligned}$$

Instruments

$$\underset{K\times 1}{\mathbf{x}_i} = x_{i1} = \cdots = x_{iM} = \text{ same instruments for each equation}$$

$$= (1, G_i, W_{2i}, TX_i, i - 1931, K_{i-1}, P_{i-1}, E_{i-1})', \;\; K = 8$$

$$\text{total parameters:} L = L_1 + L_2 + L_3 = 12,$$

$$\text{total moments:} MK = 3 \times 8 = 24$$

$$\Rightarrow MK - L = 12 \text{ overidentifying restrictions}$$

Example: Exchange Rate Regressions

$$\begin{aligned} s_{i,m} &= \alpha_m + \beta_m f_{i-1,m} + \varepsilon_{i,m} \\ i &= i, \ldots, n \text{ (months)}; m = 1, \ldots, M \text{ (currencies)} \end{aligned}$$

where

$s_{i,m}$ = log of spot exchange rate between home currency and currency m in month i

$f_{i,m}$ = log of price of forward contract for delivery of currency m in month $i+1$

Note

$$\text{cov}(\varepsilon_{i,m}, \varepsilon_{i,h}) \neq 0 \text{ due to common shocks affecting all currencies}$$

Forward Rate Unbiased Hypothesis (Uncovered Interest Rate Parity)

Under rational expectations and risk neutrality

$$\begin{aligned} E[s_{i,m}|I_{i-1}] &= f_{i-1,m},\ m = 1,\ldots,M \\ &\Rightarrow \alpha_m = 0,\ \beta_m = 1 \text{ and } E[\varepsilon_{i,m}|I_{i-1}] = 0 \end{aligned}$$

Notes:

1. Note: $\{\varepsilon_{i,m}, I_{i-1}\}$ is a MDS as a result of rational expectations

2. We would like to test all restrictions together for a more powerful test

Problem: $s_{i,m}$ and $f_{i,m}$ are nonstationary (behave like random walks) and GMM must be applied to stationary and ergodic data.

Solution: Recast UIP using stationary data

$$
\begin{aligned}
E[s_{i,m}|I_{i-1}] &= f_{i-1,m} \Rightarrow \\
E[s_{i,m}|I_{i-1}] - s_{i-1,m} &= f_{i-1,m} - s_{i-1,m} \Rightarrow \\
E[\Delta s_{i,m}|I_{i-1}] &= f_{i-1,m} - s_{i-1,m}
\end{aligned}
$$

Hence, UIP is typically tested using the so-called differences regression:

$$\begin{aligned} \Delta s_{i,m} &= \gamma_m + \delta_m (f_{i-1,m} - s_{i-1,m}) + \varepsilon_{i,m} \\ \Delta s_{i,m} &= s_{i,m} - s_{i-1,m} \end{aligned}$$

Now, UIP implies

$$\begin{aligned} \gamma_m &= 0,\ \delta_m = 1 \text{ and } E[\varepsilon_{i,m} | I_{i-1}] = 0 \\ m &= 1, \ldots, M \end{aligned}$$

SUR Model Specification in Hayashi Notation

$$y_{im} = \Delta s_{i,m}$$

$$\mathbf{z}_{im} = (1, f_{i-1,m} - s_{i,m-1})'$$

$$\boldsymbol{\delta}_m = (\gamma_m, \delta_m)', \ L_m = 2$$

$$\underset{K\times 1}{\mathbf{x}_i} = \mathbf{z}_i = \text{union of } z_{i1}, \ldots, z_{iM} = \text{ instruments for each equation}$$

$$= (1, f_{i-1,1} - s_{i-1,1}, \ldots, f_{i-1,M} - s_{i-1,M})', \ K = M + 1$$

$$\text{total parameters:} L = 2M,$$

$$\text{total moments:} MK = M(M+1)$$

$$\Rightarrow M(M+1) - 2M = M(M-1) \text{ overidentifying restrictions}$$

Sargan's Specification Test (SUR J-statistic)

$$J(\hat{\boldsymbol{\delta}}_{SUR}(\hat{\mathbf{S}}_{SUR}^{-1}), \hat{\mathbf{S}}_{SUR}^{-1})$$
$$= n\left(\underline{\mathsf{S}}_{xy} - \underline{\mathsf{S}}_{xz}\hat{\boldsymbol{\delta}}(\hat{\mathbf{S}}_{SUR}^{-1})\right)' \hat{\mathbf{S}}_{SUR}^{-1}\left(\underline{\mathsf{S}}_{xy} - \underline{\mathsf{S}}_{xz}\hat{\boldsymbol{\delta}}(\hat{\mathbf{S}}_{SUR}^{-1})\right) \sim \chi^2(M(M-1))$$

Testing UIP using GMM Wald and LR statistics

$$H_0 \quad : \quad \boldsymbol{\delta}_m = \begin{pmatrix} \gamma_m \\ \delta_m \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ m = 1, \ldots, M$$
$$H_0 \quad : \quad \underset{(2M\times 2M)}{\mathbf{R}} \underset{(2M\times 1)}{\boldsymbol{\delta}} = \underset{(2M\times 1)}{\mathbf{r}}$$

where

$$\mathbf{R} = \mathbf{I}_{2M}, \mathbf{r} = (0, 1, 0, 1, \cdots, 0, 1)'$$

Wald Statistic

$$
\begin{aligned}
Wald_{\mathsf{GMM}} &= n(\mathbf{R}\hat{\boldsymbol{\delta}} - \mathbf{r})' \left[\mathbf{R}' \widehat{\mathsf{avar}}(\hat{\boldsymbol{\delta}}) \mathbf{R}\right]^{-1} (\mathbf{R}\hat{\boldsymbol{\delta}} - \mathbf{r}) \sim \chi^2(2M) \\
\hat{\boldsymbol{\delta}} &= \hat{\boldsymbol{\delta}}(\mathbf{S}_{SUR}^{-1}) \\
\widehat{\mathsf{avar}}(\hat{\boldsymbol{\delta}}) &= \left[\underline{\underline{\mathsf{Z}}}'(\hat{\boldsymbol{\Sigma}}_{SUR}^{-1} \otimes \mathbf{I}_n)\underline{\underline{\mathsf{Z}}}\right]^{-1}
\end{aligned}
$$

LR (Difference in J) Statistic

$$
\begin{aligned}
LR_{\mathsf{GMM}} &= J(\boldsymbol{\delta}_0, \hat{\mathbf{S}}_{SUR}^{-1}) - J(\hat{\boldsymbol{\delta}}_{SUR}(\hat{\mathbf{S}}_{SUR}^{-1}), \hat{\mathbf{S}}_{SUR}^{-1}) \sim \chi^2(2M) \\
\boldsymbol{\delta}_0 &= (0, 1, 0, 1, \cdots, 0, 1)'
\end{aligned}
$$

Note: Since the null hypothesis is linear in $\boldsymbol{\delta}$, the Wald and LR statistics are numerically equivalent.

Example: Capital Asset Pricing Model (CAPM)

$$\begin{aligned} R_{im} - r_i^f &= \alpha_m + \beta_m (R_i^M - r_i^f) + \varepsilon_{im} \\ m &= 1, \ldots, M \text{ assets; } i = 1, \ldots, n \text{ months} \\ \text{cov}(\varepsilon_{im}, \varepsilon_{ih}) &\neq 0, \; \varepsilon_{im} \text{ is conditionally heteroskedastic} \end{aligned}$$

where

R_{im} = return on asset m in month i

R_i^M = return on market index in month i

r_i^f = return on risk free asset in month i

CAPM pricing relationship

$$
\begin{aligned}
E[R_{im}] - r_i^f &= \beta_m(E[R_i^M] - r_i^f) \\
&\Rightarrow \alpha_i = 0 \text{ for } m = 1, \ldots, M
\end{aligned}
$$

One test of the CAPM has null hypothesis

$$
\begin{aligned}
H_0 &: \alpha_1 = \cdots = \alpha_M = 0 \\
& \quad M \text{ zero restrictions}
\end{aligned}
$$

SUR Model Specification in Hayashi Notation

$$y_{im} = R_{im} - r_i^f$$

$$\mathbf{z}_{im} = (1, R_i^M - r_i^f)' = \text{ same for all equations!}$$

$$\boldsymbol{\delta}_m = (\alpha_m, \beta_m)', \ L_m = 2$$

$$\underset{K\times 1}{\mathbf{x}_i} = \mathbf{z}_i = \text{union of } z_{i1}, \ldots, z_{iM} = \text{ instruments for each equation}$$

$$= (1, R_i^M - r_i^f)', \ K = 2$$

$$\text{total parameters:} L = 2M$$

$$\text{total moments:} MK = 2M$$

$$\Rightarrow 2M - 2M = 0 \text{ overidentifying restrictions}$$

Sargan's Specification Test (SUR J-statistic)

$$J(\hat{\boldsymbol{\delta}}_{SUR}(\hat{\mathbf{S}}_{SUR}^{-1}), \hat{\mathbf{S}}_{SUR}^{-1})$$
$$= n\left(\underset{\sim}{\mathsf{S}}_{xy} - \underset{\sim}{\mathsf{S}}_{xz}\hat{\boldsymbol{\delta}}(\hat{\mathbf{S}}_{SUR}^{-1})\right)' \hat{\mathbf{S}}_{SUR}^{-1} \left(\underset{\sim}{\mathsf{S}}_{xy} - \underset{\sim}{\mathsf{S}}_{xz}\hat{\boldsymbol{\delta}}(\hat{\mathbf{S}}_{SUR}^{-1})\right) = 0$$

because the system is exactly identified.

Testing the CAPM

$$H_0 \; : \; \alpha_m = 0, \; m = 1, \ldots, M$$

$$H_0 \; : \; \underset{(M\times 2M)}{\mathbf{R}} \, \underset{(2M\times 1)}{\boldsymbol{\delta}} = \underset{(M\times 1)}{\mathbf{r}}$$

where

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

$$\mathbf{r} = \mathbf{0}$$