Econ 583 Lab 7

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Due: Monday, March 4.

1 Reading

1. Hayashi, Chapter 7.
2. Hall, Chapter 3.
3. My lecture notes on MLE.

2 Questions from Hayashi

1. Chapter 7, Page 500, Questions For Review #1
2. Chapter 7, Analytic Exercises, pages 502-504, #2 (information matrix equality) and 3 (trinity for linear regression).

3 Wald, LR and LM statistics based on GMM estimation

Let $X_1, \ldots, X_N$ be an iid sample of Bernoulli random variables; that is, each X_i has density $f(x; \theta_0) = \theta_0^x (1 - \theta_0)^{1-x}$. Instead of using ML estimation, consider instead estimation of θ_0 using the generalized method of moments (GMM).

1. Consider GMM estimation of θ using the single moment condition

$$g(x_t, \theta) = x_t - \theta$$

Show that $E[g(x_t, \theta)] = 0$ for $\theta = \theta_0$, and $E[g(x_t, \theta)] \neq 0$ for $\theta \neq \theta_0$. Verify that

$$\text{rank } E\left[\frac{dg(x_t, \theta_0)}{d\theta}\right] = 1$$

Additionally, show that $S = E[g(x_t, \theta_0)^2] = \theta_0(1 - \theta_0)$.

2. Using the sample moment $g_n(\theta) = n^{-1}\sum_{i=1}^{n} g(x_t, \theta)$, show that the GMM estimator of θ is $\hat{\theta}_{gmm} = n^{-1}\sum_{i=1}^{n} x_i$. What is the value of the GMM objective function $J(\hat{S}^{-1}, \hat{\theta}_{gmm}) = ng_n(\hat{\theta}_{gmm})'\hat{S}^{-1}g_n(\hat{\theta}_{gmm}) = ng_n(\hat{\theta}_{gmm})^2\hat{S}^{-1}$? How would you consistently estimate S?

3. Determine the asymptotic normal distribution for $\hat{\theta}_{gmm}$.

Next, consider testing the hypotheses $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$ using a 5 significance level.

4. Derive the Wald, LR and LM statistics for testing the above hypothesis. Because θ is a scalar, these statistics have the form

$$\begin{aligned} Wald &= n(\hat{\theta}_{gmm} - \theta_0)^2(G_n(\hat{\theta}_{gmm})^2\hat{S}^{-1}) \\ LM &= n(G_n(\theta_0)\hat{S}^{-1}g_n(\theta_0))^2(G_n(\theta_0)^2\hat{S}^{-1})^{-1} \\ LR &= J(S^{-1}, \theta_0) - J(\hat{S}^{-1}, \hat{\theta}_{gmm}) \end{aligned}$$

where $G_n(\theta) = \frac{d}{d\theta}g_n(\theta)$. For the Wald and LR statistic it is useful to estimate S using $\hat{\theta}_{gmm}$. For the LM statistic, it is useful to form an estimate of S imposing $\theta = \theta_0$.

5. Under what conditions are the GMM Wald, LM and LR statistics numerically equivalent?

4 Wald, LR and LM statistics based on maximum likelihood estimation

Let $X_1, \ldots, X_n$ be an iid sample of Bernoulli random variables; that is, each X_i has density $f(x;\theta_0) = \theta_0^x(1-\theta_0)^{1-x}$. First, consider estimation of θ_0 using maximum likelihood (ML). Let $x = (x_1, \ldots, x_n)$.

1. Derive the log-likelihood function, $\ln L(\theta|\mathbf{x})$, score function, $S(\theta|\mathbf{x}) = \frac{d}{d\theta}\ln L(\theta|\mathbf{x})$, Hessian function, $H(\theta|\mathbf{x}) = \frac{d^2}{d\theta^2}\ln L(\theta|\mathbf{x})$, and information matrix, $I(\theta_0|\mathbf{x}) = -E[H(\theta_0|\mathbf{x})] = \text{var}(S(\theta_0|\mathbf{x}))$. Verify that $E[S(\theta_0|\mathbf{x})] = 0$, and show that the ML estimate for θ is $\hat{\theta}_{mle} = n^{-1}\sum_{i=1}^{n} x_i$.

2. Determine the asymptotic normal distribution for $\hat{\theta}_{mle}$.

3. Derive the Wald, LR and LM statistics for testing the above hypothesis. These statistics have the form

$$\begin{aligned} Wald &= (\hat{\theta}_{mle} - \theta_0)^2 I(\hat{\theta}_{mle}|\mathbf{x}) \\ LM &= S(\theta_0|\mathbf{x})^2 I(\theta_0|\mathbf{x})^{-1} \\ LR &= -2[\ln L(\theta_0|\mathbf{x}) - \ln L(\hat{\theta}_{mle}|\mathbf{x})] \end{aligned}$$

Are these statistics numerically equivalent? For a 5% test, what is the decision rule to reject H_0 for each statistic?

4. Show that the Wald, LM and LR statistics have asymptotic $\chi^2(1)$ distributions. (Note: showing that LR is asymptotically chi-square involves a bit of work and some Taylor approximations)

5. How do your results compare to those that you derived for GMM in the previous question?