Econ 583
Winter 2013
HW #1

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Due: Monday, January 14

1. Recall Chebychev's inequality: Let X by any random variable with $E[X] = \mu < \infty$ and $\text{var}(X) = \sigma^2 < \infty$. Then for every $\varepsilon > 0$

$$\Pr(|X - \mu| \geq \varepsilon) \leq \frac{\text{var}(X)}{\varepsilon^2} = \frac{\sigma^2}{\varepsilon^2}$$

Suppose $X \sim N(\mu, \sigma^2)$. Using Chebychev's inequality, determine the upper bound on $\Pr(|X - \mu| \geq 3\sigma)$ and compare it to the exact bound based on the normal distribution.

2. **Cholesky decomposition**.
Result 1: For any $k \times k$ symmetric and positive semi-definite matrix $\mathbf{A}$ there exists a $k \times k$ matrix $\mathbf{B}$ such that $\mathbf{A} = \mathbf{B}\mathbf{B}'$ where $\mathbf{B}$ is a lower triangular matrix with all diagonal elements greater than or equal to zero.
Result 2: For any $k \times k$ symmetric and positive definite matrix $\mathbf{A}$ there exists a $k \times k$ matrix $\mathbf{B}$ such that $\mathbf{A} = \mathbf{B}\mathbf{B}'$ where $\mathbf{B}$ is a lower triangular matrix with all diagonal elements greater than zero.
Result 3: For any $k \times k$ symmetric and positive definite matrix $\mathbf{A}$ there exists $k \times k$ matrices $\mathbf{C}$ and $\mathbf{D}$ such that $\mathbf{A} = (\mathbf{C}\mathbf{D}^{1/2})(\mathbf{C}\mathbf{D}^{1/2})' = \mathbf{C}\mathbf{D}\mathbf{C}'$ where $\mathbf{C}$ is a lower triangular matrix with all diagonal elements equal to 1 and $\mathbf{D}$ is a diagonal matrix with positive diagonal elements.

Note that the matrices $\mathbf{A}$ and $\mathbf{B}$ have the same number of elements. Also, a simple test to see if $\mathbf{A}$ is positive definite is to compute $\mathbf{B}$ and see if all of the

diagonal elements are positive. Result 3 follows directly from Result 2 and implies that $\mathbf{B} = \mathbf{C}\mathbf{D}^{1/2}$.
(a) Let

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Verify that

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix}, \ \mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 1.5 \end{bmatrix}$$

(b) Find $\mathbf{B}$ and verify that $\mathbf{A} = \mathbf{B}\mathbf{B}'$
(c) Show that $\mathbf{A}$ can be diagonalized by computing $\mathbf{C}^{-1}\mathbf{A}\mathbf{C}^{-1\prime}$.

3. Consistency of simple estimators

Let $X_1, \ldots, X_n$ be iid random variables with $E[X_1] = \mu$ and $\text{var}(X_1) = \sigma^2 < \infty$. Consider two estimators of σ^2 :

$$\begin{aligned} \hat{\sigma}_1^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 \\ \hat{\sigma}_2^2 &= \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 \end{aligned}$$

where $\bar{X} = n^{-1}\sum_{i=1}^{n} X_i$.
(a) Compute $E[\hat{\sigma}_i^2]$ for $i = 1, 2$. You should find that $\hat{\sigma}_1^2$ is unbiased and that $\hat{\sigma}_2^2$ is biased.
(b) Compute the bias of $\hat{\sigma}_2^2$: $\text{bias}(\hat{\sigma}_2^2, \sigma^2) = E[\hat{\sigma}_2^2] - \sigma^2$. Show that $\text{bias}(\hat{\sigma}_2^2, \sigma^2) \to 0$ as $n \to \infty$.
(c) In class we showed that $\hat{\sigma}_2^2 \xrightarrow{p} \sigma^2$. Using this result, show that $\hat{\sigma}_1^2 \xrightarrow{p} \sigma^2$.

4. Hayashi, Chapter 2, page 97, Question for Review #4

5. Using Chebychev's inequality, prove Markov's LLN.

6. Hayashi, Chapter 2, page 168, Analytical Exercises #1,