Econ 583 Lec 4

Note Title 1/16/2013

Topics

- Martingale difference sequences (MDS)
- CLT for MDS
- Asymptotics for linear regression
- Finite sample and asymptotic hypothesis testing

ARCH: Example of MDS

$$u_t = z_t \sigma_t \quad , \quad z_t \sim \text{iid } N(0,1)$$

$$\sigma_t^2 = \omega + \alpha u_{t-1}^2 \qquad I_t = \{u_t, u_{t-1}, \dots\}$$

$$E[u_t | I_{t-1}] = E[z_t \sigma_t | I_{t-1}]$$

$$= \sigma_t \underbrace{E[z_t | I_{t-1}]}_{= 0}$$

$$= 0$$

$$\Rightarrow \{u_t, I_t\} \text{ is a MDS}$$

$$E[u_t^2 | I_{t-1}] = var(u_t | I_{t-1})$$
$$= E[z_t^2 \cdot \sigma_t^2 | I_{t-1}]$$
$$= \sigma_t^2 \cdot \underbrace{E[z_t^2 | I_{t-1}]}_{= 1}$$
$$= \sigma_t^2$$

We can show that u_t^2 is a serially correlated random variable

$$\sigma_t^2 = w + \alpha u_{t-1}^2$$

$$\sigma_t^2 + u_t^2 = w + \alpha u_{t-1}^2 + u_t^2$$

$$\Rightarrow \quad u_t^2 = w + \alpha u_{t-1}^2 + \underbrace{u_t^2 - \sigma_t^2}_{\xi_t}$$

$$\Rightarrow \quad u_t^2 = w + \alpha u_{t-1}^2 + \xi_t$$

$\{\xi_t, I_t\}$ is a MDS.

$\Rightarrow$ u_t^2 follows an AR(1) type process

$\Rightarrow$ u_t^2 is serially correlated.

$$y_t = x_t'\beta + \epsilon_t$$

Asymptotics for OLS

$$\hat{\beta} = \left(\Sigma_1^T x_t x_t'\right)^{-1} \Sigma_1^T x_t y_t$$

$$= \left(\Sigma_1^T x_t x_t'\right)^{-1} \Sigma_1^T x_t \left[x_t'\beta + \epsilon_t\right]$$

$$= \cancel{\left(\Sigma_1^T x_t x_t'\right)^{-1} \Sigma_1^T x_t x_t'} \beta$$

$$+ \left(\Sigma_1^T x_t x_t'\right)^{-1} \Sigma_1^T x_t \epsilon_t$$

$$\hat{\beta} = \beta + \left(\Sigma_1^T x_t x_t'\right)^{-1} \Sigma_1^T x_t \epsilon_t$$

$$\Rightarrow \quad \hat{\beta} - \beta = \underbrace{\left(\Sigma_1^T x_t x_t'\right)^{-1} \Sigma_1^T x_t \epsilon_t}_{\text{sampling error in } \hat{\beta}}$$

Show consistency of $\hat{\beta}$: $\hat{\beta} \xrightarrow{p} \beta$

or $\hat{\beta} - \beta \xrightarrow{p} 0$

Intuition: Everything is stationary & ergodic
So Ergodic Thm says sampling
converge to population moments.

$$\hat{\beta} - \beta = \left(\frac{1}{T}\sum_1^T x_t x_t'\right)^{-1} \cdot \frac{1}{T}\sum_1^T \underbrace{x_t \epsilon_t}_{g_t}$$

ergodic thm

$$\left(\frac{1}{T}\sum_1^T x_t x_t'\right)^{-1} \xrightarrow{p} \left(E[x_t x_t']\right)^{-1} = \left(\Sigma_{xx}\right)^{-1}$$

Slutsky's thm

$$\frac{1}{T}\sum_1^T x_t \epsilon_t \xrightarrow{p} E[x_t \epsilon_t] = 0$$

ergodic thm.

$$\xrightarrow{p} \Sigma_{xx}^{-1} \cdot 0 = 0$$

by Slutsky's thm.

Asymptotic Normality

$$\sqrt{T}\left(\hat{\beta} - \beta\right) = \left(\frac{1}{T}\sum_1^T x_t x_t'\right)^{-1} \frac{1}{\sqrt{T}} \sum_1^T \underbrace{x_t \epsilon_t}_{g_t}$$

$$\left(\frac{1}{T}\sum_1^T x_t x_t'\right)^{-1} \xrightarrow{p} \Sigma_{xx}^{-1}, \qquad \frac{1}{\sqrt{T}} \sum_1^T x_t \epsilon_t \xrightarrow{d} N(0, S)$$

$$\Sigma_{xx}^{-1} \cdot N(0, S)$$

Recall, if $\underset{k\times 1}{X} \sim N(\underset{k\times 1}{\mu}, \underset{k\times k}{\Sigma})$

A is a $k\times k$ matrix

$$A\cdot X \sim N(A\mu, A\Sigma A')$$

$$\Sigma_{xx}^{-1}\cdot N(0, S)$$

$$\equiv N(\underset{k\times 1}{0}, \underset{k\times k}{\Sigma_{xx}^{-1} S \Sigma_{xx}^{-1}})$$

So

$$\sqrt{T}(\hat{\beta}-\beta) \xrightarrow{d} N(0, \Sigma_{xx}^{-1} S\cdot \Sigma_{xx}^{-1})$$

or

$$\hat{\beta} \overset{A}{\sim} N\left(\beta, \frac{1}{T}\Sigma_{xx}^{-1} S \Sigma_{xx}^{-1}\right)$$

If $S = \sigma^2\cdot\Sigma_{xx}$ (i.e. $var(\epsilon_t | x_t) = \sigma^2$)

then $avar(\hat{\beta}) = \frac{1}{T}\Sigma_{xx}^{-1}\sigma^2 \cancel{\Sigma_{xx}}\,\cancel{\Sigma_{xx}^{-1}}$

$$= \frac{1}{T}\Sigma_{xx}^{-1}\cdot\sigma^2$$

Remarks

(1) $\text{avar}(\hat{\beta}) = \frac{1}{T} \Sigma_{xx}^{-1} S \Sigma_{xx}^{-1}$

= Sandwich covariance

Σ_{xx}^{-1} = bread

S = meat

(2) Typically we have to estimate Σ_{xx} and S

$\widehat{\text{avar}}(\hat{\beta}) = \frac{1}{T} \hat{\Sigma}_{xx}^{-1} \hat{S} \Sigma_{xx}^{-1}$

Where $\hat{\Sigma}_{xx} \xrightarrow{p} \Sigma_{xx}$

$\hat{S} \xrightarrow{p} S$

Q: How to estimate Σ_{xx}?

$S_{xx} = \frac{1}{T} \sum_1^T x_t x_t' \xrightarrow{p} \Sigma_{xx}$ by ergodic thm

$= \frac{1}{T} X'X$, $\underset{T \times K}{X}$ ~~matrix~~ $= \begin{bmatrix} x_{11} & \cdots & x_{1k} \\ \vdots & & \\ x_{T1} & & x_{Tk} \end{bmatrix}$

Q: How to estimate S?

$$\frac{1}{T}\sum_1^T x_t x_t' \epsilon_t^2 \xrightarrow{p} E[x_t x_t' \epsilon_t^2] = S$$

Note: $\epsilon_t = y_t - x_t'\beta$ is unobserved
b/c it depends on unknown
true β

But we observe the L.S. residual
$\hat{\epsilon}_t = y_t - x_t'\hat{\beta}$

So a natural "plug-in" type estimator
substitutes $\hat{\epsilon}_t$ for ϵ_t:

$$\hat{S} = \frac{1}{T}\sum_1^T x_t x_t' \hat{\epsilon}_t^2$$

Huber & White proved that as $n \to \infty$

$$\hat{S} \xrightarrow{p} S$$

Intuition: $\hat{\epsilon}_t = y_t - x_t'\hat{\beta}$

$$= y_t - x_t'\beta + x_t'\beta - x_t'\hat{\beta}$$

$$= \epsilon_t - x_t'(\hat{\beta} - \beta)$$

As $T \to \infty$ $\hat{\beta} \xrightarrow{p} \beta$ so $\hat{\epsilon}$ behaves like ϵ_t for large T

$$\frac{1}{T}\sum_1^T x_t x_t' \hat{\epsilon}_t^2 = \frac{1}{T}\sum_1^T x_t x_t' \epsilon_t^2 + o_p(1)$$

$$\widehat{\text{avar}}(\hat{\beta}) = \frac{1}{T}\hat{\Sigma}_{xx}^{-1}\hat{S}\hat{\Sigma}_{xx}^{-1}$$

$$= \cancel{\frac{1}{T}}\left(\cancel{\frac{1}{T}}X'X\right)^{-1}\hat{S}\left(\frac{1}{T}X'X\right)^{-1}$$

$$= T(X'X)^{-1}\hat{S}(X'X)^{-1}$$

$$\hat{S} = \frac{1}{T}\sum_1^T x_t x_t' \hat{\epsilon}_t^2$$

Look at $\hat{S} \xrightarrow{p} S$ for regression with $k=1$

$$y_t = x_t \beta + \epsilon_t$$

$$\hat{S} = \frac{1}{T} \sum_1^T x_t^2 \hat{\epsilon}_t^2 = \frac{1}{T} \sum_1^T x_t^2 \epsilon_t^2 + o_p(1)$$

$$\hat{\epsilon}_t = y_t - x_t \hat{\beta} = y_t - x_t \beta + x_t \beta - x_t \hat{\beta}$$

$$= \epsilon_t - x_t(\hat{\beta} - \beta)$$

$$\hat{\epsilon}_t^2 = \epsilon_t^2 - 2 \epsilon_t x_t (\hat{\beta} - \beta) + x_t^2 (\hat{\beta} - \beta)^2$$

$$\hat{S} = \frac{1}{T} \sum_1^T x_t^2 \hat{\epsilon}_t^2 = \frac{1}{T} \sum_1^T x_t^2 \left[\epsilon_t^2 - 2(\hat{\beta} - \beta) \epsilon_t x_t + x_t^2 (\hat{\beta} - \beta)^2 \right]$$

(1) (2)

$$= \frac{1}{T} \sum_1^T x_t \epsilon_t^2 \underbrace{- 2(\hat{\beta} - \beta) \frac{1}{T} \sum_1^T x_t^3 \epsilon_t + (\hat{\beta} - \beta)^2 \frac{1}{T} \sum_1^T x_t^4}_{o_p(1)}$$

(1) $\hat{\beta} - \beta \xrightarrow{p} 0$

So if $E[x_t^4] < \infty$ then we can justify the $o_p(1)$ terms.