Econ 583 Lec 2

Note Title 1/9/2013

Topics

- Convergence in Probability
- Laws of large numbers (LLNs)
- Slutsky theorems
- Convergence in distribution
- Central limit theorems (CLTs)

Example: Consistency of $\hat{\sigma}^2$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 \xrightarrow{p} \sigma^2 \quad \text{as } n \to \infty$$

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{p} E[X_i] = \mu \quad \text{by Chebychev's LLN}$$

Trick

$$X_i - \bar{X} = X_i - \mu + \mu - \bar{X}$$

$$(X_i - \bar{X})^2 = (X_i - \mu)^2 + 2(X_i - \mu)(\mu - \bar{X}) + (\mu - \bar{X})^2$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_1^n (X_i - \bar{X})^2 = \frac{1}{n}\sum_1^n (X_i - \mu)^2 + \frac{2}{n}(\mu - \bar{X})\sum_1^n (X_i - \mu)$$

$$+ \frac{1}{n}\sum_1^n (\mu - \bar{X})^2$$

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(3) $\frac{1}{n}\Sigma_1^n(\mu-\bar{x})^2 = \frac{n(\mu-\bar{x})^2}{n} = (\mu-\bar{x})^2$

(2) $\frac{2}{n}(\mu-\bar{x})\cdot\Sigma_1^n(X_i-\mu) = 2(\mu-\bar{x})\cdot\frac{1}{n}\Sigma_1^n(X_i-\mu)$

$= 2(\mu-\bar{x})\left[\frac{1}{n}\Sigma_1^n X_i - \frac{1}{n}\Sigma_1^n \mu\right]$

$= 2(\mu-\bar{x})[\bar{X}-\mu]$

$= -2(\mu-\bar{x})^2$

$\hat{\sigma}^2 = \frac{1}{n}\Sigma_1^n(X_i-\mu)^2 - (\mu-\bar{x})^2$

$$\underbrace{\qquad\qquad}_{Y_n} \qquad \underbrace{\qquad\qquad}_{Z_n}$$

$Z_n = -(\mu-\bar{x})^2 \xrightarrow{P} 0$

$\bar{X} \xrightarrow{P} \mu \Rightarrow \mu-\bar{X} \xrightarrow{P} 0$

Slutsky part 4 gives $-(\mu-\bar{x})^2 \xrightarrow{P} (0)^2 = 0.$

$Y_n = \frac{1}{n}\Sigma_1^n(X_i-\mu)^2$, $W_i = (X_i-\mu)^2$

$E\{W_i\} = E\{(X_i-\mu)^2\} = \sigma^2 < \infty$

And W_i are iid. So by Kolmogorov's LLN gives

$$\frac{1}{n}\sum_1^n W_i \xrightarrow{p} E\{W_i\} = \sigma^2$$

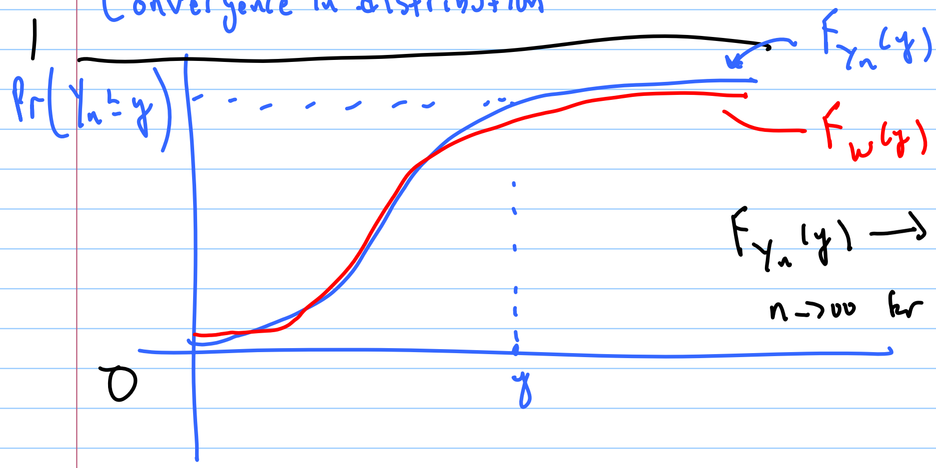
So $Y_n \xrightarrow{p} \sigma^2$. Then by Slutsky

$$Y_n + Z_n \xrightarrow{p} \sigma^2 + 0 = \sigma^2$$

If we are interested in $\hat{\sigma} = (\hat{\sigma}^2)^{1/2}$

By Slutsky $\hat{\sigma} \xrightarrow{p} (\sigma^2)^{1/2} = \sigma$ b/c $(\cdot)^{1/2}$ is continuous at $\sigma \geq 0$

Convergence in distribution

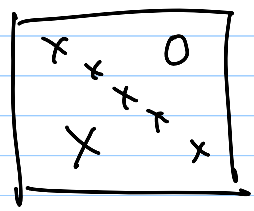


$F_{Y_n}(y) \longrightarrow F_W(y)$ as $n \to \infty$ for all continuity points

Square root factorizations of a matrix and matrix diagonalization.

$$\Sigma = \Sigma^{1/2}\Sigma^{1/2\prime}$$

where $\Sigma^{1/2}$ is lower triangular matrix with positive diagonal elements for Cholesky factorization



e.g. $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

Note: Spectral decomposition gives
$\Sigma = P\Lambda P' = (P\Lambda^{1/2})(P\Lambda^{1/2})' = \Sigma^{1/2}\Sigma^{1/2\prime}$
but $\Sigma^{1/2}$ is not lower triangular.

Goal: Standardize X to have mean vector 0 and identity covariance.

$$\Sigma = \Sigma^{1/2}\Sigma^{1/2\prime}$$

$$X \sim N(0, \Sigma)$$

$$\begin{aligned}
\mathrm{var}(\Sigma^{-1/2}X) &= \Sigma^{-1/2}\,\mathrm{var}(X)\,\Sigma^{-1/2\prime} \\
&= \Sigma^{-1/2}\,\Sigma\,\Sigma^{-1/2\prime} \\
&= \cancel{\Sigma^{-1/2}}\left(\cancel{\Sigma^{1/2}}\,\cancel{\Sigma^{1/2\prime}}\right)\cancel{\Sigma^{-1/2\prime}} \\
&= I
\end{aligned}$$

$$Z = \Sigma^{-1/2}X \sim N(0, I)$$

Therefore, if

$$X \sim N(\mu, \Sigma)$$

then

$$\underbrace{\Sigma^{-1/2}(X-\mu)}_{\text{multivariate standardization}} = Z \sim N(0, I)$$

Example: Simple regression with fixed regressors

$$\hat{\beta} = \beta + \left(\sum_i^n x_i^2\right)^{-1} \sum_i^n x_i \epsilon_i$$

$$\hat{\beta} - \beta = \left(\sum_i^n x_i^2\right)^{-1} \sum_i^n x_i \epsilon_i$$

$$\sqrt{n}(\hat{\beta} - \beta) = \left(\frac{1}{n} \sum_i^n x_i^2\right)^{-1} \frac{1}{\sqrt{n}} \sum_i^n \underbrace{x_i \epsilon_i}_{w_i}$$

$$E\{w_i\} = E\{x_i \epsilon_i\} = x_i E\{\epsilon_i\} = 0$$

$$\text{Var}(w_i) = \text{var}(x_i \epsilon_i) = x_i^2 \sigma_\epsilon^2$$

Therefore, $w_i = x_i \epsilon_i$ is not iid. It is independent but not identically distributed. Hence, we cannot use the Lindeberg-Levy CLT to derive the asymptotic distribution of the LS estimator. We need a CLT that applies to to independent but not identically distributed random variables.