Econ 583 HW # 4

Note Title 10/22/2008

Analytic Exercises

1. Hayashi Ch. 3, Questions for review pgs 215 - 216 # 7

GLS interpretation of efficient GMM

$$\underset{n\times1}{y} = \underset{n\times L}{Z}\,\underset{L\times1}{\delta} + \underset{n\times1}{\epsilon}$$

Premultiply by $\underset{K\times n}{X'}$ to give

$$\underset{K\times1}{X'y} = X'Z\delta + X'\epsilon$$

or

$$\underset{K\times1}{\tilde{y}} = \underset{K\times L}{\tilde{Z}}\,\underset{L\times1}{\delta} + \underset{K\times1}{\tilde{\epsilon}}$$

Where $\tilde{y} = X'y$, $\tilde{Z} = X'Z$ and $\tilde{\epsilon} = X'\epsilon$

Assume $var(\tilde{\epsilon}) = var(X'\epsilon) = S$.

The FGLS estimator of δ is then (see Section 1.6 of Hayashi)

$$\hat{\delta}_{FGLS} = (\tilde{Z}'\hat{S}^{-1}\tilde{Z})^{-1}\tilde{Z}'\hat{S}^{-1}\tilde{y}$$

$$= (Z'X\hat{S}^{-1}X'Z)^{-1}Z'X\hat{S}^{-1}X'y$$

$$= (\tfrac{1}{n}Z'X\,\hat{S}^{-1}\tfrac{1}{n}X'Z)^{-1}\tfrac{1}{n}Z'X\hat{S}^{-1}\tfrac{1}{n}X'y$$

$$= (S_{xz}'\hat{S}^{-1}S_{xz})^{-1}S_{xz}'\hat{S}^{-1}S_{xy}$$

$$= \hat{\delta}(\hat{S}^{-1}) = \text{efficient GMM.}$$

Hayashi, Question for review, pg 255

#1 The 2-step efficient GMM estimator is

$$\hat{\delta}(\hat{S}^{-1}) = (S_{xz}'\hat{S}^{-1}S_{xz})^{-1}S_{xz}'\hat{S}^{-1}S_{xy}$$

Where

$$\hat{S} = \tfrac{1}{n}\Sigma_1^n x_i x_i' \hat{\varepsilon}_i^2$$

$$\hat{\varepsilon}_i = y_i - z_i'\hat{\delta} \quad \text{s.t. } \hat{\delta} \xrightarrow{p} \delta$$

Under conditional homoskedasticity

$$E[x_i x_i' \epsilon_i^2] = \sigma^2 \Sigma_{xx} = S$$

$$\hat{S} \xrightarrow{p} \sigma^2 \Sigma_{xx}$$

Therefore, under conditional homoskedasticity

$$\sqrt{n}\left(\hat{\delta}(\hat{S}^{-1}) - \delta\right) = (S_{xz}' \hat{S}^{-1} S_{xz})^{-1} S_{xz}' \hat{S}^{-1} S_{x\epsilon}$$

$$\xrightarrow{d} \left(\Sigma_{xz}'(\sigma^2 \Sigma_{xx})^{-1} \Sigma_{xz}\right)^{-1} \Sigma_{xz}'(\sigma^2 \Sigma_{xx})^{-1} * N(0, \sigma^2 \Sigma_{xx})$$

$$\equiv N\left(0, (\sigma^{-2} \Sigma_{xz}' \Sigma_{xx}^{-1} \Sigma_{xz})^{-1}\right)$$

$$\equiv N\left(0, \sigma^2 (\Sigma_{xz}' \Sigma_{xx}^{-1} \Sigma_{xz})^{-1}\right)$$

#2. 2SLS is consistent when conditional heteroskedasticity does not hold.

This is b/c 2SLS is just GMM

with a particular weight matrix $\hat{W} = S_{xx}^{-1}$. So we just apply the results for GMM using an inefficient weight matrix. (prop 3.1):

$$\hat{\delta}_{2SLS} = \hat{\delta}(\hat{W}) = (S_{xz}'\hat{W}S_{xz})^{-1}S_{xz}'\hat{W}S_{xy}$$

where $\hat{W} = S_{xx}^{-1}$

From prop 3.1 $\hat{\delta}_{2SLS} \xrightarrow{p} \delta$ and

$$\sqrt{n}(\hat{\delta}_{2SLS} - \delta) \xrightarrow{d} N(0, \text{avar}(\hat{\delta}(\hat{W})))$$

where

$$\text{avar}(\hat{\delta}(\hat{W})) = (\Sigma_{xz}'W\Sigma_{xz})^{-1}\Sigma_{xz}'WSW\Sigma_{xz}(\Sigma_{xz}'W\Sigma_{xz})^{-1}$$

$$W = \text{plim}\,\hat{W} = \Sigma_{xx}^{-1}$$

$$S = E[x_i x_i' \epsilon_i^2]$$

If $\Sigma_{xx}^{-1} \neq S$ then $\hat{\delta}_{2SLS}$ is less efficient than efficient GMM.

<u>Note:</u> We can estimate $avar(\hat{\delta}_{2SLS})$ using

$$\widehat{avar}(\hat{\delta}_{2SLS}) = (S_{xz}' S_{xx}^{-1} S_{xz})^{-1} S_{xz}' S_{xx}^{-1} \hat{S}_{HC} S_{xx}^{-1} S_{xz} (S_{xz}' S_{xx}^{-1} S_{xz})^{-1}$$

$$\hat{S}_{HC} = \frac{1}{n} \Sigma_i^n x_i x_i' \hat{\varepsilon}_i^2 , \quad \hat{\varepsilon}_i = y_i - z_i' \hat{\delta}_{2SLS} .$$

Exercise 2 Asymptotic Distn of GMM LR Statistic

$y_i = z_i' \delta + \varepsilon_i$

$E[g_i] = E[x_i \varepsilon_i] = 0$

$H_0 : \delta = \delta_0$

$$LR_{GMM} = J(\delta_0, \hat{S}^{-1}) - J(\hat{\delta}(\hat{S}^{-1}), \hat{S}^{-1})$$

where

$$J(\delta_0, \hat{S}^{-1}) = n\, g_n(\delta_0)' \hat{S}^{-1} g_n(\delta_0)$$

$$J(\hat{\delta}(\hat{S}^{-1}), \hat{S}^{-1}) = n\, g_n(\hat{\delta}(\hat{S}^{-1}))' \hat{S}^{-1} g_n(\hat{\delta}(\hat{S}^{-1}))$$

$$g_n(\delta) = \frac{1}{n} \Sigma_i^n x_i \epsilon_i$$

$$= \frac{1}{n} \Sigma_i^n x_i (y_i - z_i' \delta)$$

$$= \frac{1}{n} \Sigma_i^n x_i y_i - \frac{1}{n} \Sigma_i^n x_i z_i' \delta$$

$$= S_{xy} - S_{xz} \delta$$

$$\hat{\delta}(\hat{S}^{-1}) = (S_{xz}' \hat{S}^{-1} S_{xz})^{-1} S_{xz}' \hat{S}^{-1} S_{xy}$$

(a) Show that

$$J(\delta_0, \hat{S}^{-1}) = (\sqrt{n}\, S_{x\epsilon})' \hat{S}^{-1} (\sqrt{n}\, S_{x\epsilon})$$

$$J(\hat{\delta}(\hat{S}^{-1}), \hat{S}^{-1}) = (\sqrt{n}\, S_{x\hat{\epsilon}})' \hat{S}^{-1} (\sqrt{n}\, S_{x\hat{\epsilon}})$$

Where $\hat{\epsilon}_i = y_i - z_i' \hat{\delta}(\hat{S}^{-1})$

$= \epsilon_i - z_i' (\hat{\delta}(\hat{S}^{-1}) - \delta_0)$

First, note that

$$g_n(\delta_0) = S_{xy} - S_{xz}\delta_0 = \frac{1}{n}\Sigma_i^n x_i \epsilon_i = S_{x\epsilon}$$

where $\epsilon_i = y_i - z_i' \delta_0$

Second, note that

$$g_n(\hat{\delta}(\hat{S}^{-1})) = \frac{1}{n}\Sigma_i^n x_i y_i - \frac{1}{n}\Sigma_i^n x_i z_i' \hat{\delta}(\hat{S}^{-1})$$

$$= \frac{1}{n}\Sigma_i^n x_i (y_i - z_i' \hat{\delta}(\hat{S}^{-1}))$$

$$= \frac{1}{n}\Sigma_i^n x_i \hat{\epsilon}_i = S_{x\hat{\epsilon}}$$

The desired result follows by simple substitution.

(b) Show that

$$J(\hat{\delta}(\hat{S}^{-1}), \hat{S}^{-1}) = J(\delta_0, \hat{S}^{-1}) - 2n(\hat{\delta}(\hat{S}^{-1}) - \delta_0)' S_{xz}' \hat{S}^{-1} S_{x\epsilon}$$
$$+ n(\hat{\delta}(\hat{S}^{-1}) - \delta_0)' S_{xz}' \hat{S}^{-1} S_{xz}(\hat{\delta}(\hat{S}^{-1}) - \delta_0)$$

Note that

$$S_{x\hat{\epsilon}} = \frac{1}{n}\sum_1^n x_i(y_i - z_i'\hat{\delta}(\hat{S}^{-1}))$$

$$= \frac{1}{n}\sum_1^n x_i\left(z_i'\delta_0 + \epsilon_i - z_i'\hat{\delta}(\hat{S}^{-1})\right)$$

$$= \frac{1}{n}\sum_1^n x_i\epsilon_i - \frac{1}{n}\sum_1^n x_i z_i'(\hat{\delta}(\hat{S}^{-1}) - \delta_0)$$

$$= S_{x\epsilon} - S_{xz}(\hat{\delta}(\hat{S}^{-1}) - \delta_0)$$

Then,

$$J(\hat{\delta}(\hat{S}^{-1}), \hat{S}^{-1}) = n\, S_{x\hat{\epsilon}}'\hat{S}^{-1}S_{x\hat{\epsilon}}$$

$$= n\left[S_{x\epsilon} - S_{xz}(\hat{\delta}(\hat{S}^{-1}) - \delta_0)\right]'\hat{S}^{-1}\left[S_{x\epsilon} - S_{xz}(\hat{\delta}(\hat{S}^{-1}) - \delta_0)\right]$$

$$= nS_{x\epsilon}'\hat{S}^{-1}S_{x\epsilon} - 2n\left(\hat{\delta}(\hat{S}^{-1}) - \delta_0\right)S_{xz}'\hat{S}^{-1}S_{x\epsilon}$$

$$+ n\left(\hat{\delta}(\hat{S}^{-1}) - \delta_0\right)S_{xz}'\hat{S}^{-1}S_{xz}\left(\hat{\delta}(\hat{S}^{-1}) - \delta_0\right)$$

$$= J(\delta_0, \hat{S}^{-1}) - 2n\left(\hat{\delta}(\hat{S}^{-1}) - \delta_0\right)S_{xz}'\hat{S}^{-1}S_{x\epsilon}$$

$$+ n\left(\hat{\delta}(\hat{S}^{-1}) - \delta_0\right)'S_{xz}'\hat{S}^{-1}S_{xz}\left(\hat{\delta}(\hat{S}^{-1}) - \delta_0\right)$$

(c) Using the previous result, we have

$$J(\delta_0, \hat{S}^{-1}) - J(\hat{\delta}(\hat{S}^{-1}), \hat{S}^{-1})$$

$$= 2n\left(\hat{\delta}(\hat{S}^{-1}) - \delta_0\right) S_{xz}' \hat{S}^{-1} S_{x\epsilon}$$

$$- n\left(\hat{\delta}(\hat{S}^{-1}) - \delta_0\right)' S_{xz}' \hat{S}^{-1} S_{xz} \left(\hat{\delta}(\hat{S}^{-1}) - \delta_0\right)$$

Using the result

$$\sqrt{n}\left(\hat{\delta}(\hat{S}^{-1}) - \delta_0\right) = \left(S_{xz}' \hat{S}^{-1} S_{xz}\right)^{-1} S_{xz}' \hat{S}^{-1} S_{x\epsilon}$$

We then get

$$J(\delta_0, \hat{S}^{-1}) - J(\hat{\delta}(\hat{S}^{-1}), \hat{S}^{-1}) =$$

$$2\left[\left(S_{xz}' \hat{S}^{-1} S_{xz}\right)^{-1} S_{xz}' \hat{S}^{-1} \sqrt{n} S_{x\epsilon}\right]' S_{xz}' \hat{S}^{-1} \sqrt{n} S_{x\epsilon}$$

$$- \left[\left(S_{xz}' \hat{S}^{-1} S_{xz}\right)^{-1} S_{xz}' \hat{S}^{-1} \sqrt{n} S_{x\epsilon}\right]' \cancel{S_{xz}' \hat{S}^{-1} S_{xz}}$$

$$\times \left(\cancel{S_{xz}' \hat{S}^{-1} S_{xz}}\right)^{-1} S_{xz}' \hat{S}^{-1} \sqrt{n} S_{x\epsilon}$$

$$= \left(S_{xz}' \hat{S}^{-1} \sqrt{n} S_{x\epsilon}\right)' \left(S_{xz}' \hat{S}^{-1} S_{xz}\right)^{-1} \left(S_{xz}' \hat{S}^{-1} \sqrt{n} S_{x\epsilon}\right)$$

$$\xrightarrow{d} N\left(0, \Sigma_{xz}' \underset{L\times L}{S^{-1}} \Sigma_{xz}\right)' \left(\Sigma_{xz}' \underset{L\times L}{S^{-1}} \Sigma_{xz}\right)^{-1}$$

$$* N\left(0, \Sigma_{xz}' \underset{L\times L}{S^{-1}} \Sigma_{xz}\right)$$

$$\sim \chi^2(L)$$

#3. The CU GMM estimator under conditional homoskedasticity is

$$\hat{\delta}(\hat{S}_{CU})^{-1} = \underset{\delta}{\operatorname{argmin}}\, n\, g_n(\delta)' \hat{S}_{CU}^{-1}(\delta) g_n(\delta)$$

where

$$g_n(\delta) = S_{xy} - S_{xz}\delta = \frac{1}{n}X'y - \frac{1}{n}X'Z\delta$$

$$= \frac{1}{n}X'(y - Z\delta)$$

$$\hat{S}_{CU}(\delta) = \hat{\sigma}^2(\delta) S_{xx}$$

$$= n^{-2}(y - Z\delta)'(y - Z\delta) X'X$$

Then

$$n\, g_n(\delta)' \hat{S}^{-1}(\delta) g_n(\delta) =$$

$$\cancel{n}\left(\frac{1}{\cancel{n}} X'(y - Z\delta)\right)' \left[n^{-1}(y - Z\delta)'(y - Z\delta) X'X\right]^{-1} \times \left(\frac{1}{\cancel{n}} X'(y - Z\delta)\right)$$

$$= n \frac{(y - Z\delta)' X (X'X)^{-1} X' (y - Z\delta)}{(y - Z\delta)'(y - Z\delta)}$$

$$= n \frac{(y - Z\delta)' P_X (y - Z\delta)}{(y - Z\delta)'(y - Z\delta)}$$