Econ 583 HW # 2 Delta Method Question

Note Title 11/1/2010

Q 4 (Delta Method)

$$\hat{\theta} = \begin{pmatrix} \hat{\mu} \\ \hat{\sigma}^2 \end{pmatrix} \overset{A}{\sim} N(0, V)$$

$$V = \begin{pmatrix} \sigma^2/n & 0 \\ 0 & 2\sigma^4/n \end{pmatrix}$$

$$= \text{avar}(\hat{\theta})$$

(a) We can consistently estimate avar$(\hat{\theta})$ using

$$\hat{V} = \begin{pmatrix} \hat{\sigma}^2/n & 0 \\ 0 & 2(\hat{\sigma}^2)^2/n \end{pmatrix}$$

(b) Let $q_{r_\alpha} = \mu + \sigma \cdot q^z_{r_\alpha}$, $z \sim N(0,1)$

Then

$$\Pr\left(X \leq \mu + \sigma \cdot q^z_{r_\alpha} \right)$$

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$$= \Pr\left(\frac{X-\mu}{\sigma} \le q^z_{r_\alpha} \right) = \alpha$$

$$\Rightarrow \Pr\left(X \le \underbrace{\mu + \sigma \cdot q^z_{r_\alpha}}_{q^x_{r_\alpha}} \right) = \alpha$$

(c) $q^x_{r_\alpha} = g(\theta) = \mu + \sigma \cdot q^z_{r_\alpha} = \mu + (\sigma^2)^{1/2} q^z_{r_\alpha}$

The delta method says

$$\sqrt{n}\left(g(\hat{\theta}) - g(\theta)\right) \xrightarrow{d} N\left(0, \underbrace{\frac{\partial g(\hat{\theta})'}{\partial \theta} \hat{V} \frac{\partial g(\hat{\theta})}{\partial \theta}}_{\hat{V}_g}\right)$$

Here

$$\frac{\partial g(\theta)}{\partial \mu} = 1, \quad \frac{\partial g(\theta)}{\partial \sigma^2} = \frac{1}{2}(\sigma^2)^{-\frac{1}{2}} \cdot q^z_{r_\alpha}$$

$$= \frac{1}{2}\sigma^{-1} \cdot q^z_{r_\alpha}$$

Then

$$V_g = \left(1, \tfrac{1}{2}\sigma^{-1} q^z_{r_\alpha}\right) \begin{bmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{2\sigma^4}{n} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2}\sigma^{-1} q^z_{r_\alpha} \end{bmatrix}$$

$$= \frac{\sigma^2}{n} + \frac{1}{2}\frac{\sigma^2}{n} \cdot \left(q^z_{r_\alpha}\right)^2$$

$$= \frac{\sigma^2}{n}\left[1 + \frac{1}{2}\left(q^z_{r_\alpha}\right)^2\right]$$

Therefore

$$SE\left(\hat{q}^x_{r_\alpha}\right) = \frac{\sigma}{\sqrt{n}}\left[1 + \frac{1}{2}\left(q^z_{r_\alpha}\right)^2\right]^{1/2}$$

(d) $$VaR_{0.05} = \$W \times \left[\exp\left(q^x_{r_{0.05}}\right) - 1\right]$$

$$= W\exp\left(q^x_{r_{0.05}}\right) - W$$

$$= g\left(q^x_{r_{0.05}}\right)$$

So by delta method

$$\sqrt{n}\left(g(\hat{q}^{*}_{0.05}) - g(q^{*}_{r_{0.05}})\right)$$

$$\xrightarrow{d} N\left(0, \frac{dg(\hat{q}^{*}_{r_{0.05}})}{dq^{*}_{r_{0.05}}}^2 \cdot \hat{avar}(\hat{q}^{*}_{r_{0.05}})\right)$$

$$\frac{\partial g(q^{*}_{r_{0.05}})}{\partial q^{*}_{r_{0.05}}} = W \exp(q^{*}_{r_{0.05}}) = VaR_{0.05} + W$$

Therefore

$$avar\left(\hat{VaR}_{0.05}\right) = W^2 \exp(\hat{q}^{*}_{r_{0.05}})^2 \cdot avar(\hat{q}^{*}_{r_{0.05}})$$

and so

$$\hat{SE}\left(\hat{VaR}_{0.05}\right) = W \exp(\hat{q}^{*}_{r_{0.05}}) \cdot SE(\hat{q}^{*}_{r_{0.05}})$$

(e) $SR = \dfrac{\mu - r_f}{\sigma} = (\sigma^2)^{-1/2}(\mu - r_f)$

$$\frac{\partial g(\theta)}{\partial \mu} = \sigma^{-1}, \quad \frac{\partial g(\theta)}{\partial \sigma^2} = -\frac{1}{2}(\sigma^2)^{-3/2} * (\mu - r_f)$$

$$= -\frac{1}{2}\sigma^{-3}(\mu - r_f)$$

Then avar$(g(\hat{\theta}))$ is

$$\begin{bmatrix} \sigma^{-1} & -\frac{1}{2}\sigma^{-3}(\mu-r_f) \end{bmatrix} \begin{bmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{2\sigma^4}{n} \end{bmatrix} \begin{bmatrix} \sigma^{-1} \\ -\frac{1}{2}\sigma^{-3}(\mu-r_f) \end{bmatrix}$$

$$= \begin{bmatrix} \sigma^{-1} & -\frac{1}{2}\sigma^{-3}(\mu-r_f) \end{bmatrix} \begin{bmatrix} \frac{\sigma}{n} \\ -\frac{\sigma}{n}(\mu-r_f) \end{bmatrix}$$

$$= \frac{1}{n} + \frac{1}{2}\frac{\sigma^{-2}}{n}(\mu-r_f)^2 = \frac{1}{n} + \frac{1}{2n}SR^2$$

$$= \frac{1}{n}\left[1 + \frac{1}{2}SR^2\right]$$

Then

$$SE[\hat{SR}] = \frac{1}{\sqrt{n}}\left[1 + \frac{1}{2}SR^2\right]^{1/2}$$