Economics 583: Econometric Theory I
A Primer on Asymptotics: Hypothesis Testing

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October 12, 2011

Hypothesis Testing

1. Specify hypothesis to be tested

$$H_0: \text{ null hypothesis versus. } H_1: \text{ alternative hypothesis}$$

2. Specify significance level α (size) of test

$$\alpha = \Pr(\text{Reject } H_0 | H_0 \text{ is true})$$

3. Construct test statistic, T, from observed data

4. Use test statistic T to evaluate data evidence regarding H_0

$$T \text{ is big} \Rightarrow \text{evidence against } H_0$$
$$T \text{ is small} \Rightarrow \text{evidence in favor of } H_0$$

5. Decide to reject H_0 at specified significance level if value of T falls in the rejection region

$$T \in \text{ rejection region } \Rightarrow \text{ reject } H_0$$

Remark: Usually the rejection region of T is determined by a critical value, cv_α, such that

$$\begin{aligned} T &> cv_\alpha \Rightarrow \text{reject } H_0 \\ T &\leq cv_\alpha \Rightarrow \text{ do not reject } H_0 \end{aligned}$$

Typically, cv_α is the $(1-\alpha)\times 100\%$ quantile of the probability distribution of T under H_0 such that

$$\alpha = \Pr(\text{Reject } H_0 | H_0 \text{ is true}) = \Pr(T > cv_\alpha)$$

Decision Making and Hypothesis Tests

	Reality	
Decision	H_0 is true	H_0 is false
Reject H_0	Type I error	No error
Do not reject H_0	No error	Type II error

Significance Level (size) of Test

$$\text{level} = \text{Pr}(\text{Type I error})$$
$$\text{Pr}(\text{Reject } H_0 | H_0 \text{ is true})$$

Goal: Constuct test to have a specified small significance level

$$\text{level} = 5\% \text{ or level} = 1\%$$

Power of Test

$$\begin{aligned} \text{power} &= 1 - \Pr(\text{Type II error}) \\ &= \Pr(\text{Reject } H_0 | H_0 \text{ is false}) \end{aligned}$$

or

$$\Pr(\text{Reject } H_0 | H_1 \text{ is true})$$

Goal: Construct test to have high power

Problem: Impossible to simultaneously have level ≈ 0 and power ≈ 1. As level $\longrightarrow 0$ power also $\longrightarrow 0$.

Example (Exact Tests in Finite Samples): Let $X_1, \ldots, X_n$ be iid random variables with $X_i \sim N(\mu, \sigma^2)$. Consider testing the hypotheses

$$H_0 : \mu = \mu_0 \text{ vs. } H_1 : \mu > \mu_0$$

One potential test statistic is the t-statistic

$$\begin{aligned} t_{\mu=\mu_0} &= \frac{\hat{\mu} - \mu_0}{\mathrm{SE}(\hat{\mu})} \\ \hat{\mu} &= \frac{1}{n}\sum_{i=1}^{n} X_i, \ \mathrm{SE}(\hat{\mu}) = \frac{\sigma}{\sqrt{n}} \end{aligned}$$

Intuition: We should reject H_0 if $t_{\mu=\mu_0} >> 0$.

Result: For any fixed n, under $H_0 : \mu = \mu_0$

$$t_{\mu=\mu_0} = \frac{\hat{\mu} - \mu_0}{\mathrm{SE}(\hat{\mu})} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{X_i - \mu_0}{\sigma}\right) \sim N(0, 1)$$

Remarks

1. The finite sample distribution of $t_{\mu=\mu_0}$ is $N(0,1)$, which is independent of the population parameters μ_0 and σ (sometimes called *nuisance* parameters). When a test statistic does not depend on nuisance parameters, it is called a *pivotal statistic* or *pivot*.

2. Let $\alpha \in (0,1)$ denote the significance level (size). Because we have a one-sided test, the rejection region is determined by the critical value cv_α such that

$$\Pr(\text{Reject } H_0 | H_0 \text{ is true}) = \Pr(Z > cv_\alpha) = \alpha$$

where $Z \sim N(0,1)$. Hence, cv_α is the $(1-\alpha) \times 100\%$ quantile of the standard normal distribution. For example, if $\alpha = 0.05$ then $cv_{.05} = 1.645$.

3. The t-test that rejects when $t_{\mu=\mu_0} > cv_\alpha$ is an *exact* size α finite sample test.

4. We can calculate the finite sample distribution of $t_{\mu=\mu_0}$ because we made a number of very strong assumptions: $X_1, \ldots, X_n$ be iid random variables with $X_i \sim N(\mu, \sigma^2)$. In particular, if X_i is not normally distributed then $t_{\mu=\mu_0}$ will not be normally distributed.

Example Continued: Computing Finite Sample Power

Recall,

$$\text{power} = \Pr(\text{Reject } H_0 | H_1 \text{ is true})$$

To compute power, one has to specify H_1. Suppose

$$H_1 : \mu = \mu_1 > \mu_0$$

Then

$$\text{power} = \Pr\left(\frac{\hat{\mu} - \mu_0}{\text{SE}(\hat{\mu})} > cv_\alpha \,|H_1 : \mu = \mu_1\right)$$

Under $H_1 : \mu = \mu_1$,

$$
\begin{aligned}
\frac{\hat{\mu} - \mu_0}{\mathsf{SE}(\hat{\mu})} &= \frac{\hat{\mu} - \mu_1 + \mu_1 - \mu_0}{\mathsf{SE}(\hat{\mu})} = \frac{\hat{\mu} - \mu_1}{\mathsf{SE}(\hat{\mu})} + \frac{\mu_1 - \mu_0}{\mathsf{SE}(\hat{\mu})} \\
&= Z + \delta \sim N(\delta, 1)
\end{aligned}
$$

where

$$
\begin{aligned}
Z &= \frac{\hat{\mu} - \mu_1}{\mathsf{SE}(\hat{\mu})} \sim N(0, 1) \\
\delta &= \frac{\mu_1 - \mu_0}{\mathsf{SE}(\hat{\mu})} = \sqrt{n} \left(\frac{\mu_1 - \mu_0}{\sigma} \right)
\end{aligned}
$$

Therefore, under $H_1 : \mu = \mu_1$

$$
\begin{aligned}
\text{power} &= \mathsf{Pr} \left(\frac{\hat{\mu} - \mu_0}{\mathsf{SE}(\hat{\mu})} > cv_\alpha \,|H_1 : \mu = \mu_1 \right) \\
&= \mathsf{Pr}(Z + \delta > cv_\alpha) = \mathsf{Pr}(Z > cv_\alpha - \delta)
\end{aligned}
$$

Remark:

1. Power is monotonic in δ.

2. For any fixed alternative $\mu_1 > \mu_0$, as $n \to \infty$

$$\delta = \sqrt{n}\left(\frac{\mu_1 - \mu_0}{\sigma}\right) \to \infty$$

   and power$\to$ 1.

Hypothesis Testing Based on Asymptotic Distributions

- Statistical inference in large-sample theory (asymptotic theory) is based on test statistics whose asymptotic distributions are known under the truth of the null hypothesis

- The derivation of the distribution of test statistics in large-sample theory is much easier than in finite-sample theory because we only care about the large-sample approximation to the unknown finite sample distribution
  - the LLN, CTL, Slutsky's Theorem and the Continuous Mapping Theorem (CMT) allow us to derive asymptotic distributions fairly easily

Example (Asymptotic tests). Let $X_1, \ldots, X_n$ be a sequence of covariance stationary and ergodic random variables with $E[X_i] = \mu$ and $\text{var}(X_i) = \sigma^2$. We can write

$$X_i = \mu + \varepsilon_i$$

where ε_i is a covariance stationary MDS with variance σ^2.

Consider testing the hypotheses

$$H_0 : \mu = \mu_0 \text{ vs. } H_1 : \mu \neq \mu_0$$

Two asymptotic test statistics are the asymptotic t-statistic and Wald statistic

$$\begin{aligned} t_{\mu=\mu_0} &= \frac{\hat{\mu} - \mu_0}{\widehat{\text{ase}}(\hat{\mu})}, \text{ Wald} = \left(t_{\mu=\mu_0}\right)^2 = \frac{(\hat{\mu} - \mu_0)^2}{\widehat{\text{avar}}(\hat{\mu})} \\ \hat{\mu} &= \frac{1}{n}\sum_{i=1}^{n} X_i, \ \widehat{\text{ase}}(\hat{\mu}) = \frac{\hat{\sigma}}{\sqrt{n}}, \ \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\mu})^2 \end{aligned}$$

The CLT for stationary and ergodic MDS can be used to show that under $H_0 : \mu = \mu_0$

$$\hat{\mu} \overset{A}{\sim} N\left(\mu_0, \widehat{\text{avar}}(\hat{\mu})\right), \ \widehat{\text{avar}}(\hat{\mu}) = \frac{\hat{\sigma}^2}{n}$$

The Ergodic theorem and Slutsky's theorem can be used to show that

$$t_{\mu=\mu_0} \overset{A}{\sim} N(0,1) = Z$$

Additionally, the CMT can be used to show that

$$\text{Wald} = \left(t_{\mu=\mu_0}\right)^2 \overset{A}{\sim} \chi^2(1) = \chi$$

If the significance level of each test is 5%, then the asymptotic critical values are determined by

$$\begin{aligned} \text{t-test} &: \quad \Pr(|Z| > cv_{.05}) = 0.05 \Rightarrow cv_{.05} = 1.96 \\ \text{Wald test} &: \quad \Pr(\chi > cv_{.05}) = 0.05 \Rightarrow cv_{.05} = 3.84 \end{aligned}$$

Remarks:

1. The asymptotic t-test is different from the finite sample t-test in two respects:

   (a) The way the standard error is computed ($\widehat{\text{ase}}(\hat{\mu})$ vs. $\text{se}(\hat{\mu})$)

   (b) The actual size (significance level) of the asymptotic test is 5% (the nominal significance level) only as $n \longrightarrow \infty$. In finite samples, the actual size of the asymptotic test may be smaller or larger than 5%. The difference between the actual size and the nominal size is called the *size distortion*.

Remarks continued:

2. For fixed alternatives (e.g. $H_1 : \mu = \mu_1 \neq \mu_0$) The power of the asymptotic tests converge to 1 as $n \to \infty$. That is, the asymptotic tests are *consistent* tests. To see this, consider the t-test

$$
\begin{aligned}
\frac{\hat{\mu} - \mu_0}{\widehat{\text{ase}}(\hat{\mu})} &= \sqrt{n}\left(\frac{\hat{\mu} - \mu_0}{\hat{\sigma}}\right) \\
= \sqrt{n}\left(\frac{\hat{\mu} - \mu_1 + \mu_1 - \mu_0}{\hat{\sigma}}\right) &= \sqrt{n}\left(\frac{\hat{\mu} - \mu_1}{\hat{\sigma}}\right) + \sqrt{n}\left(\frac{\mu_1 - \mu_0}{\hat{\sigma}}\right) \\
&\xrightarrow{d} N(0,1) + (\pm\infty) = \pm\infty
\end{aligned}
$$

Therefore, as $n \to \infty$

$$
\begin{aligned}
\text{power} &= \Pr(\text{reject } H_0 | H_1 \text{ is true}) \\
&= \Pr\left(\left|t_{\mu=\mu_0}\right| > 1.96\right) = \Pr(\infty > 1.96) = 1
\end{aligned}
$$

Remarks continued:

3. If there are several asymptotic tests for the same hypotheses, we can compare the *asymptotic power* of these tests by considering asymptotic power under so-called *local alternatives* of the form

$$H_1 : \mu_1 = \mu_0 + \frac{\delta}{\sqrt{n}}, \ \delta \text{ fixed}$$

Under this local alternative we can calculate non-trivial asymptotic power. To see this consider the t-test:

$$
\begin{aligned}
\frac{\hat{\mu}-\mu_0}{\widehat{\text{ase}}(\hat{\mu})} &= \sqrt{n}\left(\frac{\hat{\mu}-\mu_0}{\hat{\sigma}}\right) \\
= \sqrt{n}\left(\frac{\hat{\mu}-\mu_1+\mu_1-\mu_0}{\hat{\sigma}}\right) &= \sqrt{n}\left(\frac{\hat{\mu}-\mu_1}{\hat{\sigma}}\right) + \sqrt{n}\left(\frac{\mu_1-\mu_0}{\hat{\sigma}}\right) \\
&= \sqrt{n}\left(\frac{\hat{\mu}-\mu_1}{\hat{\sigma}}\right) + \sqrt{n}\left(\frac{\mu_0+\frac{\delta}{\sqrt{n}}-\mu_0}{\hat{\sigma}}\right) \\
&= \sqrt{n}\left(\frac{\hat{\mu}-\mu_1}{\hat{\sigma}}\right) + \frac{\delta}{\hat{\sigma}} \xrightarrow{d} N(0,1) + \frac{\delta}{\sigma} = N\left(\frac{\delta}{\sigma}, 1\right)
\end{aligned}
$$

So that

$$
\begin{aligned}
\text{power} &= \Pr(\text{reject } H_0 | H_1 \text{ is true}) \\
&= \Pr\left(\left|N\left(\frac{\delta}{\sigma}, 1\right)\right| > 1.96\right)
\end{aligned}
$$

which depends on δ.