# Econ 583 HW 6 Solutions

Note Title 12/16/2009

## 2. Asymptotics

$$y_t = \rho y_{t-1} + \epsilon_t, \quad \epsilon_t \sim WN(0, \sigma^2) \quad t = 1, \ldots, T$$

$$|\rho| < 1, \quad y_0 \text{ fixed}$$

1. Provided $|\rho| < 1$, $\{y_t\}$ is covariance stationary and ergodic.

$$E[y_t] = 0$$

$$\begin{aligned} var(y_t) &= \rho^2 var(y_{t-1}) + var(\epsilon_t) \\ &= \rho^2 var(y_t) + \sigma^2 \\ &= \frac{\sigma^2}{1-\rho^2} \end{aligned}$$

2. $\bar{y} = \frac{1}{T} \sum_1^T y_t$

$$E[\bar{y}] = \frac{1}{T} \sum_1^T E[y_t] = 0$$

By the ergodic theorem $\bar{y} \xrightarrow{p} E[y_t] = 0$

3. Because $\{y_t\}$ is covariance-stationary it has a linear process representation

$$y_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}, \quad \psi_j = \phi^j, \quad \psi_0 = 1$$

By the CLT for serially correlated linear processes

$$\sqrt{T}\,\bar{y} \xrightarrow{d} N(0, LRV)$$

$$LRV = \sum_{-\infty}^{\infty} \gamma_j = \gamma_0 + 2\sum_{j=1}^{\infty} \gamma_j$$

$$= \sigma^2 \psi(1)^2$$

For the AR(1), $\psi(L) = (1-\phi L)^{-1}$ so that $LRV = \sigma^2 (1-\phi)^{-2}$

4. A natural parametric estimate of LRV is

$$\widehat{LRV} = \hat{\sigma}^2 (1-\hat{\phi})^{-2}$$

where $\hat{\sigma}^2 \xrightarrow{p} \sigma^2$ and $\hat{\rho} \xrightarrow{p} \rho$.

For example, use the least squares estimators

$$\hat{\rho} = \left(\sum_{t=2}^{T} y_{t-1}^2\right)^{-1} \sum_{t=2}^{T} y_{t-1} y_t$$

$$\hat{\sigma}^2 = \frac{1}{T-1} \sum_t^T \hat{e}_t^2$$

$$\hat{e}_t = y_t - \hat{\rho} y_{t-1}$$

Alternatively, one could use a non-parametric estimator like the Newey-West estimator with a Bartlett kernel

5. Is $\hat{\rho}$ unbiased?

$$E[\hat{\rho}] = E\left[\left(\sum_{t=2}^{T} y_{t-1}^2\right)^{-1} \sum_{t=2}^{T} y_t y_{t-1}\right]$$

$$= \rho + E\left[\left(\sum_2^T y_{t-1}^2\right)^{-1} \sum_2^T y_{t-1} \varepsilon_t\right]$$

We cannot pass the expectation through so we cannot say that $\hat{\rho}$ is unbiased.

6. $\hat{\rho} = \left(\sum_2^T y_{t-1}^2\right)^{-1} \sum_2^T y_{t-1} y_t$

$= \left(\sum_2^T y_{t-1}^2\right)^{-1} \sum_2^T y_t (\rho y_{t-1} + \epsilon_t)$

$= \rho + \left(\sum_2^T y_{t-1}^2\right)^{-1} \sum_2^T y_{t-1} \epsilon_t$

Then

$$\hat{\rho} - \rho = \left(T^{-1} \sum_2^T y_{t-1}^2\right)^{-1} T^{-1} \sum_2^T \underbrace{y_{t-1} \epsilon_t}_{g_t}$$

Now, b/c $\{y_t\}$ is ergodic

and $\{g_t, I_t\}$ is an ergodic-stationary MDS

$$T^{-1} \sum_2^T y_{t-1}^2 \xrightarrow{p} E\{y_t^2\} = \gamma_0 = \frac{\sigma^2}{1-\rho^2}$$

$$T^{-1} \sum_2^T g_t \xrightarrow{p} E\{g_t\} = 0$$

By Slutsky theorem it follows that

$$\hat{\rho} - \rho \xrightarrow{p} 0 \Rightarrow \hat{\rho} \xrightarrow{p} \rho.$$

7. Let $g_t = y_{t-1}\epsilon_t$ and $I_t = \{y_t, \dots y_1\}$

Then $E\{g_t | I_{t-1}\} = E\{y_{t-1}\epsilon_t | I_{t-1}\}$

$$= y_{t-1} E\{\epsilon_t | I_{t-1}\} = 0$$

$\Rightarrow \{g_t, I_t\}$ is a MDS.

8. $$\sqrt{T}(\hat{\rho} - \rho) = \left(T^{-1}\Sigma_2^T y_{t-1}^2\right)^{-1} \frac{1}{\sqrt{T}} \Sigma_1^T y_{t-1}\epsilon_t$$

$$= \left(T^{-1}\Sigma_2^T y_{t-1}^2\right)^{-1} \frac{1}{\sqrt{T}} \Sigma_1^T g_t$$

where $\{g_t\}$ is an ergodic-stationary MDS

with $$E\{g_t^2\} = E[y_{t-1}^2\epsilon_t^2]$$

$$= E\{E[y_{t-1}^2\epsilon_t^2 | I_{t-1}]\}$$

$$= E\{y_{t-1}^2 E[\epsilon_t^2 | I_{t-1}]\}$$

$$= \sigma^2 E[y_{t-1}^2]$$

$$= \sigma^2\gamma_0 = \sigma^2(\sigma^2/(1-\rho^2))$$

By the CLT for ergodic-stationary MDS

$$\frac{1}{\sqrt{T}} \sum_t^T g_t \xrightarrow{d} N(0, \sigma^2 \gamma_0)$$

Then by Slutsky's theorem

$$\sqrt{T}(\hat{\rho} - \rho) \xrightarrow{d} \gamma_0^{-1} \cdot N(0, \sigma^2 \gamma_0)$$

$$\equiv N(0, \sigma^2 \gamma_0^{-1})$$

$$\equiv N(0, \cancel{\sigma^2}[\cancel{\sigma^2}(1-\rho^2)]^{-1})$$

$$\equiv N(0, (1-\rho^2))$$

Note: $\text{avar}(\sqrt{T}(\hat{\rho}-\rho)) = (1-\rho^2)$

9. A simple estimate for $\text{avar}(\sqrt{T}(\hat{\rho}-\rho))$ is $\widehat{\text{avar}}(\sqrt{T}(\hat{\rho}-\rho)) = 1 - \hat{\rho}^2$

where $\hat{\rho}$ is the least squares estimate of $\rho$.

Nonlinear GMM

$Y_1, \ldots, Y_n$ iid Gamma$(\alpha, \beta)$:

$$f(y|\theta) = \frac{\alpha^\beta}{\Gamma(\beta)} e^{-\alpha y} y^{\beta-1}, \quad y > 0$$

$$\alpha > 0, \ \beta > 0$$

Moments: $E[Y] = \beta/\alpha$

$$E[Y^2] = \beta(\beta+1)/\alpha^2$$

$$E[\ln Y] = \psi(\beta) - \ln \alpha, \quad \psi(\beta) = \frac{d}{d\beta} \ln \Gamma(\beta)$$

$$E[1/Y] = \alpha(\beta-1)$$

1. Let $w_t = \{Y_t, Y_t^2, \ln Y_t, 1/Y_t\}$. Then define $\theta = (\alpha, \beta)'$ and

$$\underset{4 \times 1}{g(w_t, \theta)} = \begin{bmatrix} Y_t - \beta/\alpha \\ Y_t^2 - \beta(\beta+1)/\alpha^2 \\ \ln Y_t - \psi(\beta) + \ln \alpha \\ 1/Y_t - \alpha/(\beta-1) \end{bmatrix}$$

Under the true model $g(w_t, \theta_0) = 0$

which can be verified directly given the moment equations

2. Because $Y_1, \ldots, Y_n$ is a random sample $g(w_t, \theta)$ is ergodic-stationary.

3. For efficient GMM using $g(w_t, \theta)$, the GMM objective function is

$$J(\theta, \hat{S}^{-1}) = g_n(\theta)' \hat{S}^{-1} g_n(\theta)$$

Where $g_n(\theta) = \frac{1}{n} \sum_i^n g(w_t, \theta)$

$$\hat{S} = \frac{1}{n} \sum_1^n g(w_t, \hat{\theta}) g(w_t, \hat{\theta})'$$

$$\hat{\theta} \xrightarrow{p} \theta.$$

For example, $\hat{\theta}$ could be computed from an inefficient GMM estimation with $\hat{W} = I_4$:

$$\hat{\theta} = \underset{\theta}{\arg\min}\; g_n(\theta)' I_n g_n(\theta)$$

The F.O.C's for minimizing $J(\theta, \hat{S}^{-1})$ are

$$\underset{2\times 1}{0} = \frac{\partial J}{\partial \theta} = \underset{2\times 4}{\frac{\partial g_n(\hat{\theta})'}{\partial \theta'}} \underset{4\times 4}{\hat{S}^{-1}} \underset{4\times 1}{g_n(\hat{\theta})}$$

Given the moment equations this produces a system of non-linear equations in $\alpha, \beta$ and no analytic solution exists. The GN iteration scheme is

$$\hat{\theta}_{n+1} = \hat{\theta}_n + (G_n' \hat{S}^{-1} G_n)^{-1} G_n' \hat{S}^{-1} g_n(\hat{\theta}_n)$$

where $\underset{4\times 2}{G_n} = \dfrac{\partial g_n(\hat{\theta}_n)}{\partial \theta'}$