Econ 583 Hw 5 Solutions

Note Title 12/16/2009

Asymptotics for AR(1)

$$y_t = \rho y_{t-1} + \epsilon_t, \quad \epsilon_t \sim WN(0, \sigma^2) \quad t = 1, \dots, T$$

$$|\rho| < 1, \quad y_0 \text{ fixed}$$

1. Provided $|\rho| < 1$, $\{y_t\}$ is covariance stationary and ergodic.

$$E[y_t] = 0$$

$$\begin{aligned} var(y_t) &= \rho^2 var(y_{t-1}) + var(\epsilon_t) \\ &= \rho^2 var(y_t) + \sigma^2 \\ &= \frac{\sigma^2}{1-\rho^2} \end{aligned}$$

2. $\bar{y} = \frac{1}{T} \sum_1^T y_t$

$$E[\bar{y}] = \frac{1}{T} \sum_1^T E[y_t] = 0$$

By the ergodic theorem $\bar{y} \xrightarrow{p} E[y_t] = 0$

3. Because $\{y_t\}$ is covariance-stationary it has a linear process representation

$$y_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}, \quad \psi_j = \phi^j, \quad \psi_0 = 1$$

By the CLT for serially correlated linear processes

$$\sqrt{T}\,\bar{y} \xrightarrow{d} N(0, LRV)$$

$$LRV = \sum_{-\infty}^{\infty} \gamma_j = \gamma_0 + 2\sum_{j=1}^{\infty} \gamma_j$$

$$= \sigma^2 \psi(1)^2$$

For the AR(1), $\psi(L) = (1-\phi L)^{-1}$ so that $LRV = \sigma^2 (1-\phi)^{-2}$

4. A natural parametric estimate of LRV is

$$\widehat{LRV} = \hat{\sigma}^2 (1-\hat{\phi})^{-2}$$

where $\hat{\sigma}^2 \xrightarrow{p} \sigma^2$ and $\hat{\rho} \xrightarrow{p} \rho$.

For example, use the least squares estimators

$$\hat{\rho} = \left(\sum_{t=2}^{T} y_{t-1}^2\right)^{-1} \sum_{t=2}^{T} y_{t-1} y_t$$

$$\hat{\sigma}^2 = \frac{1}{T-1} \sum_{t}^{T} \hat{e}_t^2$$

$$\hat{e}_t = y_t - \hat{\rho}\, y_{t-1}$$

Alternatively, one could use a non-parametric estimator like the Newey-West estimator with a Bartlett kernel

5. Is $\hat{\rho}$ unbiased?

$$E[\hat{\rho}] = E\left[\left(\sum_{t=2}^{T} y_{t-1}^2\right)^{-1} \sum_{t=2}^{T} y_t y_{t-1}\right]$$

$$= \rho + E\left[\left(\sum_{2}^{T} y_{t-1}^2\right)^{-1} \sum_{2}^{T} y_{t-1} \varepsilon_t\right]$$

We cannot pass the expectation through so we cannot say that $\hat{\rho}$ is unbiased.

6. $\hat{\rho} = \left(\Sigma_2^T y_{t-1}^2\right)^{-1} \Sigma_2^T y_{t-1} y_t$

$= \left(\Sigma_2^T y_{t-1}^2\right)^{-1} \Sigma_2^T y_t (\rho y_{t-1} + \epsilon_t)$

$= \rho + \left(\Sigma_2^T y_{t-1}^2\right)^{-1} \Sigma_2^T y_{t-1} \epsilon_t$

Then

$$\hat{\rho} - \rho = \left(T^{-1} \Sigma_2^T y_{t-1}^2\right)^{-1} T^{-1} \Sigma_2^T \underbrace{y_{t-1} \epsilon_t}_{g_t}$$

Now, b/c $\{y_t\}$ is ergodic

and $\{g_t, I_t\}$ is an ergodic-stationary MDS

$$T^{-1} \Sigma_2^T y_{t-1}^2 \xrightarrow{p} E\{y_t^2\} = \gamma_0 = \frac{\sigma^2}{1-\rho^2}$$

$$T^{-1} \Sigma_2^T g_t \xrightarrow{p} E\{g_t\} = 0$$

By Slutsky theorem it follows that

$$\hat{\rho} - \rho \xrightarrow{p} 0 \implies \hat{\rho} \xrightarrow{p} \rho.$$

7. Let $g_t = y_{t-1}\epsilon_t$ and $I_t = \{y_t, \dots y_1\}$

Then $E\{g_t \mid I_{t-1}\} = E\{y_{t-1}\epsilon_t \mid I_{t-1}\}$

$= y_{t-1} E\{\epsilon_t \mid I_{t-1}\} = 0$

$\Rightarrow \{g_t, I_t\}$ is a MDS.

8. $\sqrt{T}(\hat{\rho} - \rho) = \left(T^{-1}\Sigma_2^T y_{t-1}^2\right)^{-1} \frac{1}{\sqrt{T}} \Sigma_1^T y_{t-1}\epsilon_t$

$= \left(T^{-1}\Sigma_2^T y_{t-1}^2\right)^{-1} \frac{1}{\sqrt{T}} \Sigma_1^T g_t$

where $\{g_t\}$ is an ergodic-stationary MDS

with $E\{g_t^2\} = E[y_{t-1}^2 \epsilon_t^2]$

$= E\{E[y_{t-1}^2 \epsilon_t^2 \mid I_{t-1}]\}$

$= E\{y_{t-1}^2 E[\epsilon_t^2 \mid I_{t-1}]\}$

$= \sigma^2 E[y_{t-1}^2]$

$= \sigma^2 \gamma_0 = \sigma^2 (\sigma^2/(1-\rho^2))$

By the CLT for ergodic-stationary MDS

$$\frac{1}{\sqrt{T}} \Sigma_2^T g_t \xrightarrow{d} N(0, \sigma^2 \gamma_0)$$

Then by Slutsky's theorem

$$\sqrt{T}(\hat{\rho} - \rho) \xrightarrow{d} \gamma_0^{-1} \cdot N(0, \sigma^2 \gamma_0)$$

$$\equiv N(0, \sigma^2 \gamma_0^{-1})$$

$$\equiv N(0, \cancel{\sigma^2} [\cancel{\sigma^2}(1-\rho^2)]^{-1})$$

$$\equiv N(0, (1-\rho^2))$$

Note: $\text{avar}(\sqrt{T}(\hat{\rho} - \rho)) = (1-\rho^2)$

9. A simple estimate for $\text{avar}(\sqrt{T}(\hat{\rho} - \rho))$ is $\widehat{\text{avar}}(\sqrt{T}(\hat{\rho} - \rho)) = 1 - \hat{\rho}^2$

where $\hat{\rho}$ is the least squares estimate of ρ.