Econ 583 HW 1 Solutions

Note Title 10/6/2011

Q 1: Let X be a rv with $E[X] = \mu$ and $Var(X) = \sigma^2 < \infty$. Then Chebychev's inequality says

$$\Pr\{|X-\mu| \geq \epsilon\} \leq \frac{Var(X)}{\epsilon^2} = \frac{\sigma^2}{\epsilon^2}$$

Suppose $X \sim N(\mu, \sigma^2)$. Then Chebychev's inequality says

$$\Pr\{|X-\mu| \geq \overbrace{3\sigma}^{\epsilon}\} \leq \frac{\sigma^2}{9\sigma^2} = \frac{1}{9}$$

Because $X \sim N(\mu, \sigma^2)$

$$\Pr\{|X-\mu| \geq 3\sigma\}$$

$$= \Pr\left\{\left|\frac{X-\mu}{\sigma}\right| \geq 3\right\} = 1 - \Pr\{-3 \leq Z \leq 3\}$$

for $Z \sim N(0,1)$. Using the pnorm() function in R we get

$$1 - \Pr\{-3 \leq Z \leq 3\} = 1 - 0.9973 = 0.0027$$

Econ 583 HW #1 Solutions for Problem 3

Note Title 10/18/2009

Consistency of Simple estimators

$x_1, \dots, x_n$ iid with $E[x] = \mu$, $var(x) = \sigma^2 < \infty$

$\hat{\sigma}_1 = \frac{1}{n-1} \Sigma_i^n (x_i - \bar{x})^2$, $\bar{x} = \frac{1}{n} \Sigma_i^n x_i$

$\hat{\sigma}_2 = \frac{1}{n} \Sigma_i^n (x_i - \bar{x})^2$

(a) Compute $E[\hat{\sigma}_i^2]$ $i = 1, 2$

$\hat{\sigma}_1^2 = (n-1)^{-1} \underset{\sim}{\epsilon}' M_1 \underset{\sim}{\epsilon}$, $\epsilon_i = x_i - \bar{x}$

and $\epsilon = (\epsilon_1, \dots, \epsilon_n)$, $M_1 = I_n - P_1$

and $P_1 = \underline{1}(\underline{1}'\underline{1})^{-1}\underline{1}'$ with $\underline{1} = (1, \dots, 1)'$.

Then

$$E[\underset{\sim}{\epsilon}' M_1 \underline{\epsilon}] = E\left[tr\left(\overbrace{\underline{\epsilon}' M_1 \underline{\epsilon}}^{\text{scalar}}\right)\right]$$

$$= E\left[tr\left(M_1 \underset{\sim}{\epsilon}\underset{\sim}{\epsilon}'\right)\right] \quad \left(tr(AB) = tr(BA)\right)$$

$tr(\cdot)$ & $E[\cdot]$ are linear operators

$$= tr\left(E\left[M_1 \underset{\sim}{\epsilon}\underset{\sim}{\epsilon}'\right]\right)$$

$$= tr\left(M_1 E[\epsilon\epsilon']\right)$$

$$= tr\left(M_1 \sigma^2 I_n\right)$$

$$= \sigma^2 tr(M_1) = \sigma^2 rank(M_1)$$

$$= \sigma^2 (n-1) \quad \text{b/c } M_1 \text{ is Idempotent}$$

Hence, $E[\hat{\sigma}_1^2] = (n-1)^{-1} E[\underset{\sim}{\epsilon}' M_1 \underset{\sim}{\epsilon}]$

$$= (n-1)^{-1}(n-1)\sigma^2 = \sigma^2$$

Next, $\hat{\sigma}_v^2 = \frac{n-1}{n} \cdot \frac{1}{n-1} \Sigma_1^n (x_i - \bar{x})^2$

$$= \frac{n-1}{n} \hat{\sigma}_1^2$$

$$\Rightarrow E[\hat{\sigma}_v^2] = \frac{n-1}{n} E[\hat{\sigma}_1^2] = \frac{n-1}{n}\sigma^2$$

(b) $bias(\hat{\sigma}_v^2, \sigma^2) = E[\hat{\sigma}_v^2] - \sigma^2$

$$= \frac{n-1}{n}\sigma^2 - \sigma^2 = \left(\frac{n-1}{n} - 1\right)\sigma^2$$

$$= -\frac{1}{n}\sigma^2$$

Clearly, bias$(\hat{\sigma}_2^2, \sigma_2) \to 0$ as $n \to \infty$.

(c) $\hat{\sigma}_1^2 = \frac{n}{n-1} \hat{\sigma}_2^2$. As $n \to \infty$

$$\frac{n}{n-1} \longrightarrow 1 \quad \text{and} \quad \hat{\sigma}_2^2 \xrightarrow{p} \sigma^2$$

By Slutsky, $\hat{\sigma}_1^2 \xrightarrow{p} 1 \cdot \sigma^2 = \sigma^2$.

. Hayashi, ch 2, pg. 97 #4

Suppose $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N(0, \sigma^2)$.

Does it follow that $\hat{\theta}_n \xrightarrow{p} \theta$?

Write $\quad \hat{\theta}_n - \theta = \frac{1}{\sqrt{n}} \sqrt{n} (\hat{\theta}_n - \theta)$

Then $\quad \frac{1}{\sqrt{n}} \longrightarrow 0 \quad , \sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N(0, \sigma^2)$

as $n \to \infty$. Therefore by Slutsky

$$\hat{\theta}_n - \theta \xrightarrow{d} 0 \cdot N(0, \sigma^2) = 0$$

Hence $\hat{\theta}_n \xrightarrow{p} \theta$.

5. Recall Markov's WLN: $X_1, \dots, X_n$ uncorrelated with $E[X_i] = \mu_i < \infty$ and $var(X_i) = \sigma_i^2 \leq M < \infty \ \forall i = 1, \dots, n$.

Then $\bar{X} - \bar{\mu} = \frac{1}{n}\sum_i^n X_i - \frac{1}{n}\sum_i^n \mu_i$

$= \frac{1}{n}\sum_i^n (X_i - \mu_i) \xrightarrow{?} 0$

Chebychev's inequality says for any r.v X with $E[X] = \mu$ and $var(X) = \sigma^2 < \infty$

$$Pr(|X - \mu| > \epsilon) \leq \frac{var(X)}{\epsilon^2}$$

Let $X = \bar{X}$ and $\mu = \bar{\mu}$. Then

$$Pr(|\bar{X} - \bar{\mu}| > \epsilon) \leq \frac{var(\bar{X})}{\epsilon^2}$$

Now,

$$\operatorname{var}(\bar{x}) = \operatorname{var}\left(\frac{1}{n}\Sigma_1^n x_i\right) = \frac{1}{n^2}\Sigma_1^n \operatorname{var}(x_i)$$

$$= \frac{1}{n^2}\Sigma_1^n \sigma_i^2 < \frac{n M}{n^2} = \frac{M}{n}$$

Therefore,

$$\Pr(|\bar{x} - \bar{\mu}| > \epsilon) \leq \frac{M}{n} \to 0$$

as $n \to \infty$. $\blacksquare$

6. Hayashi, ch 2, pg. 168 # 1

$$z_n = \begin{cases} 0 & \text{with prob } (n-1)/n \\ n^2 & \text{"} \quad 1/n \end{cases}$$

Show that $\operatorname{plim}_{n\to\infty} z_n = 0$ but

$$\lim_{n\to\infty} E[z_n] \neq 0$$

For any $\epsilon > 0$

$$\Pr\left(|Z_n| > \epsilon \right) = \frac{1}{n} \xrightarrow[n \to \infty]{} 0$$

Hence $Z_n \xrightarrow{P} 0$. However,

$$E\{Z_n\} = 0 \cdot \frac{(n-1)}{n} + n^2 \cdot \frac{1}{n} = n$$

$$\to \infty \quad \text{as } n \to \infty.$$ $\blacksquare$