The Multivariate Regression Model

The SUR model doesn't require that the X_i matrices be different. They can have common components and they can have all of the same components. If each X_i is the same, we have what is called the <u>multivariate regression model</u>:

$$\underset{N\times 1}{\underset{\sim}{y}_i} = \underset{N\times K}{X}\ \underset{K\times 1}{\underset{\sim}{\beta}_i} + \underset{N\times 1}{\underset{\sim}{\epsilon}_i} \qquad X_i = X \qquad i=1,\dots,M$$

The giant regression model becomes

$$\underset{MT\times 1}{\begin{bmatrix} \underset{\sim}{y}_1 \\ \underset{\sim}{y}_2 \\ \vdots \\ \underset{\sim}{y}_M \end{bmatrix}} = \underset{MT\times MK}{\begin{bmatrix} X & 0 & & \\ 0 & X & & \\ & & \ddots & \\ & & & X \end{bmatrix}} \underset{MK\times 1}{\begin{bmatrix} \underset{\sim}{\beta}_1 \\ \underset{\sim}{\beta}_2 \\ \vdots \\ \underset{\sim}{\beta}_M \end{bmatrix}} + \underset{MT\times 1}{\begin{bmatrix} \underset{\sim}{\epsilon}_1 \\ \underset{\sim}{\epsilon}_2 \\ \vdots \\ \underset{\sim}{\epsilon}_M \end{bmatrix}}$$

or

$$\underset{MT\times 1}{\tilde{y}} = (I_M \otimes X)\underset{MK\times 1}{\tilde{\beta}} + \underset{MT\times 1}{\tilde{\epsilon}} \qquad (*)$$

where

$$\tilde{\beta} = (\tilde{\beta}_1', \ldots, \tilde{\beta}_M')'$$

$$\tilde{\epsilon} = (\tilde{\epsilon}_1', \tilde{\epsilon}_2', \ldots, \tilde{\epsilon}_M')$$

$$E\{\tilde{\epsilon}\tilde{\epsilon}'\} = \Sigma \otimes I_T = V.$$

Because of the common value of X, the OLS estimator of β in $(*)$ simplies:

$$\hat{\tilde{\beta}}_{OLS} = (X'V^{-1}X)^{-1}X'V^{-1}y$$

$$= (\cancel{X}(I_m \otimes X)'(\Sigma \otimes I_T)^{-1}(I_m \otimes X))^{-1}(I_m \otimes X)'(\Sigma \otimes I_T)^{-1}$$

$$= ((I_m \otimes X')(\Sigma^{-1} \otimes I_T)(I_m \otimes X))^{-1}(I_m \otimes X')(\Sigma^{-1} \otimes I_T)$$

$$= ((\Sigma^{-1} \otimes X')(I_m \otimes X))^{-1}(\Sigma^{-1} \otimes X')y$$

$$= (\Sigma^{-1} \otimes X'X)^{-1}(\Sigma^{-1} \otimes X')y$$

$$= (\Sigma \otimes (X'X)^{-1})(\Sigma^{-1} \otimes X')y$$

$$= (I_m \otimes (X'X)^{-1}X')y$$

$$= \begin{pmatrix} (X'X)^{-1}X'y_1 \\ (X'X)^{-1}X'y_2 \\ \vdots \\ (X'X)^{-1}X'y_m \end{pmatrix}$$

$$= \begin{pmatrix} \hat{\beta}_{1,OLS} \\ \hat{\beta}_{2,OLS} \\ \vdots \\ \hat{\beta}_{m,OLS} \end{pmatrix} \longleftarrow \text{equation by equation OLS.}$$

$$= \hat{\underset{\sim}{\beta}}_{OLS}.$$

<u>Result</u> (Zellner 1962): OLS = GLS in SUR model if $X_i = X$ $i = 1, .., M$.

<u>Application</u>: CAPM regression system

$$R_{it} - r_{Ft} = \alpha_i + \beta_i (R_{Mt} - r_{Ft}) + \epsilon_{it} \quad \begin{matrix} i = 1, .., M \\ t = 1, .., T \end{matrix}$$

SUR model with $y_{it} = R_{it} - r_{Ft}$, $x_{it} = (1, R_{Mt} - r_{Ft})'$ $\forall i$

$\Rightarrow$ OLS equation by equation is efficient.

Hypothesis Testing in the SUR Model

Under standard regularity conditions

$$\underset{K\times 1}{\hat{\underset{\sim}{\beta}}_{FGLS}},\ \hat{\underset{\sim}{\beta}}_{IFGLS} \overset{A}{\sim} N\Big(\underset{K\times 1}{\underset{\sim}{\beta}},\ \underbrace{(X'(\hat{\Sigma}^{-1}\otimes I_T)X)^{-1}}_{K\times K}\Big)$$

where $K = \sum_{i=1}^{M} k_i$

The asymptotic normality of $\hat{\beta}_{FGLS}$ allows for the construction of Walds tests for hypotheses about β.

For example, consider an hypothesis of the form

$$H_0:\ \underset{J\times K}{R}\ \underset{K\times 1}{\underset{\sim}{\beta}} = \underset{J\times 1}{\underset{\sim}{q}}$$

which imposes J linear restrictions on β. For example, in the foreign exchange regression ~~the~~ Rational expectations and risk neutrality imply the restrictions

$$\underset{2M\times 1}{\underset{\sim}{\beta}} = \begin{bmatrix} \underset{\sim}{\beta}_1 \\ \underset{\sim}{\beta}_2 \\ \vdots \\ \beta_M \end{bmatrix} = \begin{bmatrix} \binom{0}{1} \\ \binom{0}{1} \\ \vdots \\ \binom{0}{1} \end{bmatrix}$$

Hence

$$\underset{\sim}{R} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ 0 & 0 & \cdots & & 1 \end{bmatrix} = I_{2M}$$

and

$$\underset{2M \times 1}{\underset{\sim}{q}} = \begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \vdots \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \Rightarrow 2M \text{ restrictions.}$$

The Wald test takes the form

$$W = (R\underset{\sim}{\hat{\beta}}_{FULS} - \underset{\sim}{q})' \, \hat{var}(R\hat{\beta}_{FULS})^{-1} (R\hat{\beta}_{FULS} - q)$$

where $\hat{var}(R\hat{\beta}_{FULS}) = R'(X'(\hat{\Sigma}^{-1} \otimes I_T)X)^{-1}R$

Under $H_0: R\underset{\sim}{\beta} = \underset{\sim}{q}$

$$W \overset{A}{\sim} \chi^2(J)$$

A Simple LM test for H_0: Σ diagonal
(Greene pg. 681)

In the SUR model, OLS equation by equation is efficient provided the X's are exogenous and the error covariance matrix is diagonal. Therefore, it is of interest to test to see if $E[\underset{\sim}{\epsilon}\underset{\sim}{\epsilon}'] = V$ is diagonal. Since

$$V = \Sigma \otimes I_T$$

where

$$\underset{M \times M}{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1M} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2M} \\ \vdots & & & \\ \sigma_{1M} & & \cdots & \sigma_M^2 \end{pmatrix}$$

is diagonal if Σ is diagonal we wish to test

$$H_0: \underset{M \times M}{\Sigma} \text{ is diagonal} = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \sigma_2^2 & \\ 0 & & \ddots \; \sigma_M^2 \end{bmatrix}$$

$\Rightarrow$ $\dfrac{M(M-1)}{2}$ restrictions on Σ (all off diagonal elements are zero)

Breusch e Pagan (1980) develop a simple LM test in a model where the errors are normally distributed. The advantage of the LM test is that under the null, Σ is diagonal, and so OLS equation by equation is efficient. Hence the test statistic can be computed easily using OLS residuals from each equation.

OLS equation by equation gives

$$\underset{\sim}{y}_i = X_i \underset{\sim}{\hat{\beta}}_{i,OLS} + \underset{\sim}{\hat{\epsilon}}_{i,OLS} \qquad i=1,\ldots,M$$

Now form the $M(M-1)/2$ values of

$$\hat{\rho}_{ij} = \frac{\hat{\sigma}_{ij}}{(\hat{\sigma}_i^2 \hat{\sigma}_j^2)^{1/2}} = \text{estimated cross equation correlation}$$

where $\hat{\sigma}_{ij} = \frac{1}{T} \underset{\sim}{\hat{\epsilon}}_i{}' \underset{\sim}{\hat{\epsilon}}_j = \frac{1}{T}\sum_{t=1}^{T} \hat{\epsilon}_{it}\hat{\epsilon}_{jt}$.

Under H_0: Σ diagonal $\Rightarrow \rho_{ij} = 0 \quad j, i = 1, \ldots, M, \; i \neq$

The LM test statistic is

$$LM = T \cdot \sum_{i=2}^{M} \sum_{j=1}^{i-1} \hat{\rho}_{ij}^2$$

and under H_0 it can be shown that

$$LM \overset{A}{\sim} \chi^2\left(\frac{M(M-1)}{2}\right)$$

# of restrictions under H_0.

Reject H_0 : Σ diagonal at 5% level if

$$LM > \chi^2_{0.95}\left(\frac{M(M-1)}{2}\right)$$