Maximum likelihood Estimation of SUR model

The SUR model is

$$y_{it} = \underset{1\times k_i}{\underset{\sim}{x}_{it}'} \underset{k_i\times 1}{\underset{\sim}{\beta}_i} + \epsilon_{it} \qquad \begin{array}{l} t=1,\dots,T \\ i=1,\dots,M. \end{array}$$

t'th obs on i'th equation.

Let $\underset{\sim}{x}_t^*$ ~~t~~ denote the $k^*\times 1$ vector of unique elements of $\underset{\sim}{x}_{1t}, \underset{\sim}{x}_{2t}, \dots, \underset{\sim}{x}_{mt}$. If all of the x_{it}s are unique then $\underset{\sim}{x}_t^* = (\underset{\sim}{x}_{1t}', \underset{\sim}{x}_{2t}', \dots, \underset{\sim}{x}_{mt}')'$ and the dimension of $\underset{\sim}{x}_t^*$ is $K\times 1$ where $K = \sum_{i=1}^{M} k_i$.

If some of the $\underset{\sim}{x}_{it}$'s have elements in common (e.g. a vector of 1's) then $\underset{\sim}{x}_t^*$ ~~[illegible]~~ does not include these redundant variables and the dimension of $\underset{\sim}{x}_t^*$ is less than $K = \sum_{i=1}^{M} k_i$.

The likelihood function under normal errors is simplified if we reparameterize the SUR model as follows: ~~Thes~~

$$y_{it} = \underset{1\times k^*}{\underset{\sim}{x_{it}^{*\prime}}} \underset{k^*\times 1}{\underset{\sim}{\pi_i}} + \epsilon_{it} \qquad \begin{matrix} i=1,\ldots,M \\ t=1,\ldots,T \end{matrix}$$

where π_i = coeffs on x_t^* from ith equation.
The structure of the SUR model generally places zero restrictions on π_i

Example: Foreign exchange regressions

$$s_{i,t+1} = \alpha_i + \gamma_i f_{i,t} + \epsilon_{i,t+1}$$

or

$$s_{i,t} = \alpha_i + \gamma_i f_{i,t-1} + \epsilon_{i,t} \qquad \begin{matrix} i=1,\ldots,M \\ t=1,\ldots,T. \end{matrix}$$

where s_{it} = log of spot exchange rate & f_{it} = log of forward exchange rate. Here

$$\underset{1\times 2}{x_{it}} = (1, f_{i,t-1})' \qquad i=1,\ldots,M$$

so that only variable in common is ~~the~~ 1. Hence

$$\underset{(M+1)\times 1}{x_{it}^*} = (1, f_{1,t-1}, f_{2,t-1}, \ldots, f_{M,t-1})'$$

and

$$\underset{(M+1)\times 1}{\underset{\sim}{\Pi_1}} = \begin{pmatrix} \alpha_1 \\ \gamma_1 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, \quad \underset{(M+1)\times 1}{\underset{\sim}{\Pi_2}} = \begin{pmatrix} \alpha_2 \\ 0 \\ \gamma_2 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}, \ldots, \quad \underset{(M+1)\times 1}{\underset{\sim}{\Pi_M}} = \begin{pmatrix} \alpha_M \\ 0 \\ 0 \\ \vdots \\ 0 \\ \gamma_M \end{pmatrix}$$

Next, stack the equations horizontally via

$$\underbrace{\begin{bmatrix} y_{1t} & y_{2t} & \cdots & y_{Mt} \end{bmatrix}}_{1\times M} = \underset{1\times K^*}{\underset{\sim}{x_t^{*'}}} \underbrace{\begin{bmatrix} \underset{K^*\times 1}{\underset{\sim}{\Pi_1}} & \underset{K^*\times 1}{\underset{\sim}{\Pi_2}} & \cdots & \underset{K^*\times 1}{\Pi_M} \end{bmatrix}}_{K^*\times M}$$

$$+ \underbrace{\begin{bmatrix} \epsilon_{1t} & \epsilon_{2t} & \cdots & \epsilon_{Mt} \end{bmatrix}}$$

or

$$\underset{1\times M}{\underset{\sim}{y_t'}} = \underset{\sim}{x_t^{*'}} \underset{K^*\times M}{\Pi} + \underset{1\times M}{\underset{\sim}{\epsilon_t'}} \qquad t = 1, \ldots, T$$

or

$$\underset{M\times 1}{\underset{\sim}{y_t}} = \underset{M\times K^*}{\Pi'} \underset{K^*\times 1}{\underset{\sim}{x_t^*}} + \underset{M\times 1}{\underset{\sim}{\epsilon_t}} \qquad t = 1, \ldots, T.$$

In this reparameterization

$$E\left[\underset{\sim}{\epsilon_t}\underset{\sim}{\epsilon_t'}\right] = \underset{M\times M}{\Sigma}$$

Assuming

$$\underset{M\times 1}{\underset{\sim}{\epsilon_t}} \sim \text{iid } N(0, \underset{M\times M}{\Sigma})$$

the joint density for $\underset{\sim}{y_t}$ takes the form

$$f(\underset{\sim}{y_t}; \Pi, \Sigma) = (2\pi)^{-M/2} |\Sigma|^{-1/2} \exp\left\{-\frac{1}{2} \underset{\sim}{\epsilon_t}' \Sigma^{-1} \underset{\sim}{\epsilon_t}\right\}$$

$$= (2\pi)^{-M/2} |\Sigma|^{-1/2} \exp\left\{-\frac{1}{2} (\underset{\sim}{y_t} - \Pi' \underset{\sim}{x_t^*})' \Sigma^{-1} (\underset{\sim}{y_t} - \Pi' \underset{\sim}{x_t^*})\right\}$$

The likelihood function for the sample is then

$$L(\Pi, \Sigma | Y, X^*) = \prod_{t=1}^{T} f(y_t; \Pi, \Sigma)$$

$$= (2\pi)^{-MT/2} |\Sigma|^{-T/2} \exp\left\{-\frac{1}{2} \sum_1^T (\underset{\sim}{y_t} - \Pi' \underset{\sim}{x_t^*})' \Sigma^{-1} (\underset{\sim}{y_t} - \Pi' \underset{\sim}{x_t}\right.$$

and the log-likelihood is

$$\ln L(\Pi, \Sigma) = -\frac{MT}{2} \ln(2\pi) - \frac{T}{2} \ln |\Sigma| - \frac{1}{2} \sum_{t=1}^{T} (y_t - \Pi' x_t^*)' \Sigma^{-1} (y_t - \Pi'$$

Remarks

(1) Usually there are zero restrictions on Π so that the MLE does not have a simple closed form solution (MLE imposes these restrictions)

(2) If Π is unrestricted then it is straightforward to show that the MLE for Π is equivalent to OLS equation by equation on

$$y_i = X^* \underset{\sim}{\pi_i} + \underset{\sim}{\epsilon_i}$$

(Recall, OLS = GLS if $X_i = X \ \forall i$)

(3) The log-likelihood function can be analytically concentrated w.r.t. Σ giving

$$\ln L^c(\Pi) = \frac{-TM}{2}\left(\ln(2\pi)+1\right) - \frac{T}{2}\ln|\Sigma(\Pi)|$$

where
$$\Sigma(\Pi) = \frac{1}{T}\sum_{t=1}^{T} e_t e_t'$$
$$= \frac{1}{T}\sum_{t=1}^{T}(y_t - \Pi' x_t^*)(y_t - \Pi' x_t^*)'$$

Hence, the MLE for Π solves

$$\min_{\Pi} \ln \left| \Sigma(\Pi) \right|$$

$$= \min_{\Pi} \ln \left| \frac{1}{T} \Sigma_1^T (y_t - \Pi' x_t^*)(y_t - \Pi' x_t^*)' \right|$$

~~and it can be~~ subject to any zero restrictions on Π.

It can be shown that the MLE is numerically identical to iterated feasible GLS on the SUR model.

The MLE for Σ is then

$$\hat{\Sigma}_{MLE} = \Sigma(\hat{\Pi}_{MLE})$$

(4) LR tests have a particularly simple form. For example, consider testing the k^* restrictions.

$$H_0: \Pi = \Pi_0 \quad \text{vs.} \quad \Pi \neq \Pi_0$$

The unrestricted log-likelihood is

$$\ln L^c(\hat{\Pi}_{MLE}) = -\frac{TM}{2}(\ln(2\pi)+1) - \frac{T}{2}\ln|\Sigma(\hat{\Pi}_{MLE})|$$

and the restricted log-likelihood is

$$\ln L^c(\Pi_0) = -\frac{TM}{2}(\ln(2\pi+1)) - \frac{T}{2}\ln|\Sigma(\Pi_0)|$$

so that

$$LR = -2\left[\ln L^c(\Pi_0) - \ln L^c(\hat{\Pi}_{MLE})\right]$$

$$= T\left(\ln|\Sigma(\Pi_0)| - \ln|\Sigma(\hat{\Pi}_{MLE})|\right)$$

and under $H_0: \Pi = \Pi_0$

$$LR \overset{A}{\sim} \chi^2(k^*)$$