Systems of Regression Equations

General set up

$$\underset{T\times 1}{y_1} = \underset{T\times K_1}{X_1}\,\underset{K_1\times 1}{\beta_1} + \underset{T\times 1}{\epsilon_1} \qquad \epsilon_1 \sim (0, \sigma_1^2 I_T)$$

$$\underset{T\times 1}{y_2} = \underset{T\times K_2}{X_2}\,\underset{K_2\times 1}{\beta_2} + \underset{T\times 1}{\epsilon_2} \qquad \epsilon_2 \sim (0, \sigma_2^2 I_T)$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$\underset{T\times 1}{y_M} = \underset{T\times K_M}{X_M}\,\underset{K_M\times 1}{\beta_M} + \underset{T\times 1}{\epsilon_M} \qquad \epsilon_M \sim (0, \sigma_M^2 I_T)$$

- X_i is exogenous $i = 1, \ldots, M$ (Not a simultaneous equations system
- elements of ϵ_i may be contemporaneously correlated with the elements of ϵ_j
- Variables in X_i may overlap with those in X_j (common regressors)
- Context may imply across equation restrictions among the βs & the elements of the across equation covariances

Examples

I. Exchange Rate Regressions

$$s_{i,t+1} = \alpha_i + \gamma_i f_{it} + \epsilon_{i,t+1} \qquad i = 1, \dots, M \text{ currenc} \quad t = 1, \dots, T \text{ months}$$

where

$s_{i,t}$ = log of spot exchange rate for currency i (wrt \$ say) in month t

$f_{i,t}$ = log of forward exchange rate for forward contract that settles in month $t+1$ (1-month forward contract) for currency i in month t

In the previous notation

$$y_{it} = s_{i,t+1}$$

$$\underset{\sim}{x}_{it} = (1, f_{i,t})' \quad \longleftarrow \text{ different regressors for each equation.}$$

$$\underset{\sim}{\beta}_i = (\alpha_i, \gamma_i)'$$

Also $\epsilon_{i,t}$ is likely to be correlated with ϵ_{jt} due to common shocks affecting exchange rates

Under rational expectations and risk neutrality

it can be shown that

$$E[S_{i,t+1} / I_t] = f_{it} \qquad i = 1, \ldots, M$$

↑ info available at time t

$$\Rightarrow \quad \alpha_i = 0 \quad \text{and} \quad \beta_i = 1 \qquad i = 1, \ldots, M$$

$$E\{\epsilon_{it+1} | I_t] = 0$$

Hence testing the RE & Risk neutrality assumption requires testing across equation restrictions.

2. CAPM Regressions

The Capital asset pricing model (CAPM) can be represented in the excess return market model regression

$$R_{it} - r_{Ft} = \alpha_i + \beta_i (R_{Mt} - r_{Ft}) + \epsilon_{it}$$

$i = 1, \ldots, M$ assets

$t = 1, \ldots, T$ months

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where

R_{it} = continuously compounded return on asset i in month t $\left(R_{it} = \ln\left(\frac{P_{it}}{P_{it-1}}\right)\right)$

r_{Ft} = return on risk free rate (30 day T-bill)

R_{Mt} = return on Market index

(usually value weighted index of traded equities)

Here

$y_{it} = (R_{it} - r_{Ft})$ $\quad i = 1, .., M$

$\underset{\sim}{x}_{it} = (1, R_{Mt} - r_{Ft})'$ $\quad i = 1, .., M$

Same regressors for each equation!

$\underset{\sim}{\beta}_i = (\alpha_i, \beta_i)'$

ε_{it} may be correlated with ε_{jt}

but ε_{it} not correlated with ε_{js} for $t \neq s$.

CAPM requires $\alpha_i = 0$ $\quad i = 1, .., M$ $\left\{ \text{cross equation restriction} \right.$

$\Rightarrow E[R_{it} - r_{Ft} \mid R_{Mt}]$

$= \beta_i (R_{Mt} - r_{Ft})$

$\Rightarrow \beta_i$ captures portfolio risk

3. Factor Demand Functions

General Form based on minimizing cost function

$$\underset{\substack{\uparrow \\ \text{input demand}}}{X_i} = f_i(\underset{\substack{\uparrow \\ \text{output/}}}{Y}, \underset{\substack{\nwarrow \\ \text{input prices}}}{P}; \beta) + \epsilon_i \qquad i = 1,.., M$$

Economic theory places a lot of restrictions on the coefficients β.

Estimating the Multi equation System

Consider the general "seemingly unrelated regression system

$$\underset{\cancel{T}\times 1}{\underset{\sim}{y}_i} = \underset{\cancel{T}\times K_i}{X_i} \underset{K_i \times 1}{\underset{\sim}{\beta}_i} + \underset{\cancel{T}\times 1}{\underset{\sim}{\epsilon}_i} \qquad i = 1,..,M$$

where

$$E[\epsilon_{it}] = 0 \qquad i = 1,..,M, \; t = 1,..,N$$

but

$$E\{\epsilon_{it}^2\} = \sigma_i^2 \qquad i = 1,.., M$$

and

$$E[\epsilon_{it}\epsilon_{jt}] = \sigma_{ij} \qquad i,j = 1,.., M$$

$$E[\epsilon_{it}\epsilon_{js}] = 0 \qquad i,j = 1,.., M; \; t \neq s$$

→ Initially assume that there are no cross equation restrictions ←

We can stack ~~eqe~~ each equation and create a "giant" regression model

$$\underset{MT\times 1}{\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \vdots \\ \tilde{y}_M \end{bmatrix}} = \underset{MT\times K}{\begin{bmatrix} X_1 & & & \\ & X_2 & & \\ & & \ddots & \\ & & & X_M \end{bmatrix}} \underset{K\times 1}{\begin{bmatrix} \tilde{\beta}_1 \\ \tilde{\beta}_2 \\ \vdots \\ \tilde{\beta}_M \end{bmatrix}} + \underset{MT\times 1}{\begin{bmatrix} \tilde{\epsilon}_1 \\ \tilde{\epsilon}_2 \\ \vdots \\ \tilde{\epsilon}_M \end{bmatrix}}$$

where

$$K = \sum_{i=1}^{M} k_i$$

or

$$\underset{MT\times 1}{\tilde{y}} = \underset{MT\times K}{X} \underset{K\times 1}{\tilde{\beta}} + \underset{MT\times 1}{\tilde{\epsilon}}$$

Note

$$E[\tilde{\epsilon}\tilde{\epsilon}'] = E\left[\begin{bmatrix} \tilde{\epsilon}_1 \\ \tilde{\epsilon}_2 \\ \vdots \\ \tilde{\epsilon}_M \end{bmatrix} (\tilde{\epsilon}_1' \; \tilde{\epsilon}_2' \; \cdots \; \tilde{\epsilon}_M')\right]$$

$$= E \begin{bmatrix} \epsilon_1 \epsilon_1' & \epsilon_1 \epsilon_2' & \cdots & \epsilon_1 \epsilon_M' \\ \epsilon_2 \epsilon_1' & \epsilon_2 \epsilon_2' & \cdots & \epsilon_2 \epsilon_M' \\ & & \ddots & \\ \epsilon_M \epsilon_1' & & \cdots & \epsilon_M \epsilon_M' \end{bmatrix}$$

$$= \underset{MT \times MT}{\begin{bmatrix} \sigma_1^2 I_T & \sigma_{12} I_T & \cdots & \sigma_{1M} I_T \\ \sigma_{12} I_T & \sigma_{22} I_T & \cdots & \sigma_{2M} I_T \\ & & \ddots & \\ & & & \sigma_M^2 I_T \end{bmatrix}} = \underset{MT \times MT.}{V}$$

which is not a diagonal Matrix. Hence OLS is not, in general, the most efficient estimation procedure. GLS is the efficient estimator and has the familar form

$$\hat{\beta}_{GLS} = (X'V^{-1}X)^{-1} X'V^{-1}y$$

Digression on Kronecker products

$$\underset{n\times k}{A} \otimes \underset{m\times L}{B} = \begin{bmatrix} a_{11}\cdot B & a_{12}\cdot B & \cdots & a_{1k}\cdot B \\ \vdots & & & \\ a_{n1}\cdot B & a_{n2}\cdot B & \cdots & a_{nk}\cdot B \end{bmatrix}$$

$(nm)\times(kL)$

e.g. $$\begin{bmatrix} 3 & 0 \\ 5 & 2 \end{bmatrix} \otimes \begin{bmatrix} 1 & 4 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 3\cdot\begin{bmatrix} 1 & 4 \\ 4 & 7 \end{bmatrix} & 0\cdot\begin{bmatrix} 1 & 4 \\ 4 & 7 \end{bmatrix} \\ 5\cdot\begin{bmatrix} 1 & 4 \\ 4 & 7 \end{bmatrix} & 2\cdot\begin{bmatrix} 1 & 4 \\ 4 & 7 \end{bmatrix} \end{bmatrix}$$

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Properties of kronecker products

(1) $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

(2) $\text{trace}(A \otimes B) = \text{tr}(A)\cdot\text{tr}(B)$

(3) $|A \otimes B| = |A|^m \, |B|^n$

(4) $(A \otimes B)' = A' \otimes B'$

(5) $(A \otimes B)(C \otimes D) = AC \otimes BD$

(6) $A \otimes (C+D) = A\otimes C + A \otimes D$

Using kronecker products we may write

$$V = E\{\underset{\sim}{\epsilon}\underset{\sim}{\epsilon}'\} = \begin{bmatrix} \sigma_1^2 I_T & \sigma_{12} I_T & \cdots & \sigma_{1m} I_T \\ & \sigma_{22} I_T & & \\ & & \ddots & \\ & & & \sigma_{m}^2 I_T \end{bmatrix}$$

$$= \underbrace{\underset{m \times m}{\Sigma} \otimes \underset{T \times T}{I_T}}_{mT \times mT}$$

where $$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1m} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2m} \\ & & \ddots & \\ & & & \sigma_m^2 \end{pmatrix}$$

In addition

$$V^{-1} = (\Sigma \otimes I_T)^{-1}$$

$$= (\Sigma^{-1} \otimes I_T)$$

so that

$$\hat{\beta}_{GLS} = (X'(\Sigma^{-1} \otimes I_T)X)^{-1} X'(\Sigma^{-1} \otimes I_T)\underset{\sim}{y}$$

Feasible GLS

The GLS estimator is infeasible if Σ is unknown (which is usually the case). In practice we must estimate Σ giving rise to the feasible GLS estimator

$$\hat{\underset{\sim}{\beta}}_{FGLS} = \left(X'(\hat{\Sigma}^{-1} \otimes I_T)X\right)^{-1} X'(\hat{\Sigma}^{-1} \otimes I_T)\underset{\sim}{y}$$

where

$$\hat{\Sigma} = \begin{pmatrix} \hat{\sigma}_1^2 & \hat{\sigma}_{12} & \cdots & \hat{\sigma}_{1M} \\ & \hat{\sigma}_2^2 & \cdots & \hat{\sigma}_{2M} \\ & & \ddots & \\ & & & \hat{\sigma}_M^2 \end{pmatrix}$$

is a consistent estimator of Σ, i.e. $\hat{\Sigma} \xrightarrow{p} \Sigma$.

Estimating Σ

~~σ_{ij} can be estimated from~~

Since X_i are exogenous, β_i can be estimated consistently (but not efficiently) by OLS on each equation

$$\underset{T\times 1}{\underset{\sim}{y}_i} = \underset{T\times k_i}{X_i}\, \underset{k_i\times 1}{\hat{\underset{\sim}{\beta}}_i} + \underset{T\times 1}{\hat{\epsilon}_i} \qquad i=1,\dots,M$$

Furthermore, σ_{ij} can be consistently estimated using

$$\hat{\sigma}_{ij} = \frac{1}{T}\, \hat{\underset{\sim}{\epsilon}}_i{}' \hat{\underset{\sim}{\epsilon}}_j \qquad i,j=1,\dots,M$$

$$= \frac{1}{T} \sum_{t=1}^{T} \hat{\epsilon}_{it}\hat{\epsilon}_{jt}$$

<u>Iterated FGLS</u>

The estimate of Σ in FGLS uses the inefficient OLS estimates of β. The iterated FGLS estimator repeats the construction of the FGLS estimator using an updated estimator of Σ based on the FGLS estimates:

$$\hat{\hat{\Sigma}} = \begin{bmatrix} \hat{\hat{\sigma}}_1^2 & \hat{\hat{\sigma}}_{12} & \cdots & \hat{\hat{\sigma}}_{1M} \\ & \ddots & & \\ & & & \hat{\hat{\sigma}}_M^2 \end{bmatrix}$$

where

$$\hat{\sigma}_{ij} = \frac{1}{T}\hat{\epsilon}_i'\hat{\epsilon}_j$$

$$\hat{\epsilon}_i = y_i - X_i\hat{\beta}_{i,FGLS}$$

$$\hat{\epsilon}_j = y_j - X_j\hat{\beta}_{j,FGLS}.$$

This process gives

$$\hat{\beta}_{IFGLS} = (X'(\hat{\Sigma}^{-1}\otimes I_T)X)^{-1}(X'(\hat{\Sigma}\otimes I_T)y)$$

and can be "iterated" until $\hat{\beta}_{IFGLS}$ doesn't change from iteration to iteration.

<u>Result</u>: Iterated FGLS is numerically equivalent to ~~the~~ MLE if ~~ε or Normality~~

$$\underset{MT\times 1}{\underset{\sim}{\epsilon}} \sim N(0, \Sigma\otimes I_T)$$