Economics 582
Random Effects Estimation

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Random Effects Model

$$\begin{aligned} y_{it} &= \mathbf{x}'_{it}\boldsymbol{\beta} + \alpha_i + \eta_{it} \\ E[\mathbf{x}_{it}\alpha_i] &= 0 \text{ (no endogeneity)} \\ E[\alpha_i|\mathbf{x}_{it}] &= \alpha \end{aligned}$$

Hence, the model can be re-written as

$$\begin{aligned} y_{it} &= \alpha + \mathbf{x}'_{it}\boldsymbol{\beta} + u_i + \eta_{it} \\ E[\mathbf{x}_{it}u_i] &= 0 \\ E[u_i|\mathbf{x}_{it}] &= 0 \end{aligned}$$

RE framework is a SUR model with common coefficients - efficient estimation depends on the covariance structure of $u_i + \eta_{it}$

The Homoskedastic Equi-correlation Framework

A common covariance structure in the RE model is based on the assumptions

$$u_i \sim \text{iid } (0, \sigma_u^2)$$
$$\eta_{it} \sim \text{iid } (0, \sigma_\eta^2)$$

Let $\varepsilon_{it} = u_i + \eta_{it}$. Then

$$cov(\varepsilon_{it}, \varepsilon_{is}) = \begin{cases} \sigma_u^2 & \text{for } t \neq s \\ \sigma_u^2 + \sigma_\eta^2 & \text{for } t = s \end{cases}$$

and

$$cor(\varepsilon_{it}, \varepsilon_{is}) = \frac{\sigma_u^2}{\sigma_u^2 + \sigma_\eta^2} \text{ for } t \neq s$$

Note,

$$cov(\varepsilon_{it}, \varepsilon_{js}) = 0 \text{ for } i \neq j$$

In matrix notation we have

$$\begin{aligned}
\underset{T\times 1}{\mathbf{y}_i} &= \mathbf{W}_i\boldsymbol{\delta} + \boldsymbol{\varepsilon}_i \\
\mathbf{W}_i &= [\mathbf{1}, \mathbf{X}_i],\ \boldsymbol{\delta} = (\alpha, \boldsymbol{\beta}')' \\
\boldsymbol{\varepsilon}_i &= u_i\mathbf{1}_T + \boldsymbol{\eta}_i \\
E[\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i'] &= \boldsymbol{\Omega}
\end{aligned}$$

Then

$$\begin{aligned}
E[\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i'] &= E[(u_i\mathbf{1}_T + \boldsymbol{\eta}_i)(u_i\mathbf{1}_T + \boldsymbol{\eta}_i)'] \\
&= E[u_i^2]\mathbf{1}_T\mathbf{1}_T' + E[\boldsymbol{\eta}_i\boldsymbol{\eta}_i'] \\
&= \sigma_u^2\mathbf{1}_T\mathbf{1}_T' + E[\boldsymbol{\eta}_i\boldsymbol{\eta}_i']
\end{aligned}$$

Now

$$E[\boldsymbol{\eta}_i\boldsymbol{\eta}_i'] = \begin{pmatrix} \sigma_\eta^2 & & \\ & \ddots & \\ & & \sigma_\eta^2 \end{pmatrix} = \sigma_\eta^2 \mathbf{I}_T$$

Hence,

$$\begin{aligned} E[\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i'] &= \sigma_u^2 \mathbf{1}_T\mathbf{1}_T' + \sigma_\eta^2 \mathbf{I}_T \\ &= \boldsymbol{\Omega} \\ &= \begin{pmatrix} \sigma_u^2+\sigma_\eta^2 & \sigma_u^2 & \cdots & \sigma_u^2 \\ \sigma_u^2 & \sigma_u^2+\sigma_\eta^2 & \cdots & \sigma_u^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_u^2 & \sigma_u^2 & \cdots & \sigma_u^2+\sigma_\eta^2 \end{pmatrix} \end{aligned}$$

Remarks

- Because $\mathbf{\Omega} \neq \sigma_\varepsilon^2 \mathbf{I}_T$ pooled OLS is not an efficient estimator.

- The efficient estimator is pooled GLS, which is also the SUR estimator

Efficient Estimation of the RE model

Using $\mathbf{\Omega}^{-1} = \mathbf{\Omega}^{-1/2}\mathbf{\Omega}^{-1/2\prime}$, the transformed model is

$$\mathbf{\Omega}^{-1/2}\mathbf{y}_i = \mathbf{\Omega}^{-1/2}\mathbf{W}_i\boldsymbol{\delta} + \mathbf{\Omega}^{-1/2}\boldsymbol{\varepsilon}_i$$

or

$$\begin{aligned} \tilde{\mathbf{y}}_i &= \tilde{\mathbf{W}}_i\boldsymbol{\delta} + \tilde{\boldsymbol{\varepsilon}}_i \\ E[\tilde{\boldsymbol{\varepsilon}}_i\tilde{\boldsymbol{\varepsilon}}_i'] &= \mathbf{I}_T \end{aligned}$$

Hence, the GLS (SUR) estimator is the pooled OLS estimator

$$\begin{aligned} \hat{\boldsymbol{\delta}}_{RE} &= \left(\sum_{i=1}^{n} \tilde{\mathbf{W}}_i'\tilde{\mathbf{W}}_i\right) \sum_{i=1}^{n} \tilde{\mathbf{W}}_i'\tilde{\mathbf{y}}_i \\ &= \left(\sum_{i=1}^{n} \mathbf{W}_i'\mathbf{\Omega}^{-1}\mathbf{W}_i\right) \sum_{i=1}^{n} \mathbf{W}_i'\mathbf{\Omega}^{-1}\mathbf{y}_i \end{aligned}$$

Feasible GLE estimation

Because σ_u^2 and σ_η^2 are typically unknown, GLS is not feasible. The feasible GLS estimator is

$$\hat{\boldsymbol{\delta}}_{RE} = \left(\sum_{i=1}^{n} \mathbf{W}_i' \hat{\boldsymbol{\Omega}}^{-1} \mathbf{W}_i \right) \sum_{i=1}^{n} \mathbf{W}_i' \hat{\boldsymbol{\Omega}}^{-1} \mathbf{y}_i$$

where

$$\hat{\boldsymbol{\Omega}} = \hat{\sigma}_u^2 \mathbf{1}_T \mathbf{1}_T' + \hat{\sigma}_\eta^2 \mathbf{I}_T$$
$$\hat{\sigma}_u^2 \xrightarrow{p} \sigma_u^2 \text{ and } \hat{\sigma}_\eta^2 \xrightarrow{p} \sigma_\eta^2$$

Q: How to consistently estimate σ_u^2 and σ_η^2?

Consistent Estimation of σ_u^2 and σ_η^2

There are several ways to consistently estimate σ_u^2 and σ_η^2. The most common estimators are as follows.

- Estimation of σ_η^2 is based on the FE Within estimator from the regression

$$\begin{aligned}\tilde{y}_{it} &= \tilde{\mathbf{x}}_{it}'\boldsymbol{\beta} + \tilde{\eta}_{it} \\ \tilde{y}_{it} &= y_{it} - \bar{y}_i,\ \tilde{\mathbf{x}}_{it} = \mathbf{x}_{it} - \bar{x}_i,\ \tilde{\eta}_{it} = \eta_{it} - \bar{\eta}_i\end{aligned}$$

Then

$$\hat{\sigma}_\eta^2 = \frac{1}{n(T-1)-L}\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{\tilde{\eta}}_{it}^2 \xrightarrow{p} \sigma_\eta^2$$

- Estimation of σ_u^2 is bsed on the Between regression

$$\begin{aligned}\bar{y}_i &= \bar{\mathbf{w}}_i'\boldsymbol{\delta} + u_i + \bar{\eta}_i,\ i = 1, \ldots, n \\ var(u_i + \bar{\eta}_i) &= \sigma_u^2 + \frac{\sigma_\eta^2}{T},\ \bar{\eta}_i = \frac{1}{T}\sum_{t=1}^{T}\eta_{it}\end{aligned}$$

Then

$$\hat{\sigma}_u^2 = \frac{1}{n-(L+1)}\sum_{i=1}^{n}\left(\bar{y}_i - \bar{\mathbf{w}}_i'\hat{\boldsymbol{\delta}}\right)^2 - \frac{1}{T}\hat{\sigma}_\eta^2 \xrightarrow{p} \sigma_u^2$$

Alternative Form of RE Estimator

Result: The FGLS estimator

$$\hat{\boldsymbol{\delta}}_{RE} = \left(\sum_{i=1}^{n} \mathbf{W}_i' \hat{\boldsymbol{\Omega}}^{-1} \mathbf{W}_i \right) \sum_{i=1}^{n} \mathbf{W}_i' \hat{\boldsymbol{\Omega}}^{-1} \mathbf{y}_i$$

can be computed as the pooled OLS estimator from the transformed model

$$\begin{aligned} y_{it} - \hat{\lambda} \bar{y}_i &= (\mathbf{w}_{it} - \hat{\lambda} \bar{\mathbf{w}}_i)' \boldsymbol{\delta} + v_{it} \\ v_{it} &= (1 - \hat{\lambda}) u_i + (\eta_{it} - \hat{\lambda} \bar{\eta}_i) \end{aligned}$$

and λ is a consistent estimator of

$$\lambda = 1 - \frac{\sigma_\eta}{\sqrt{T\sigma_u^2 + \sigma_\eta^2}}$$

Remarks:

- $\hat{\lambda} = 0 \Rightarrow \hat{\boldsymbol{\delta}}_{RE} =$ pooled OLS

- $\hat{\lambda} = 1 \Rightarrow \hat{\boldsymbol{\delta}}_{RE} = \hat{\boldsymbol{\beta}}_{RE} = \hat{\boldsymbol{\beta}}_{FE}$

- $\hat{\lambda} \to 1$ as $T \to \infty$

Linear Panel Example: Hours and Wages (Cameron and Trivedi, ch. 21)

- Does labor supply respond to wages?
  - For people working, result is ambiguous due to offsetting substitution and income effects

- Data on $N = 532$ males for each of the $T = 10$ years from 1979 to 1988 (5320 total obvs)

- Simple linear panel regression

$$\begin{aligned} \ln hrs_{it} &= \beta \ln wages_{it} + \alpha_i + \eta_{it} \\ & \eta_{it} \text{ is indep over } i \text{ but not } t \end{aligned}$$

	POLS	**Between**	**Within**	**First Diff**	**RE**
α	7.442	7.483			7.346
β	.083	.067	.168	.109	.119
Panel Robust SE(β)	(.030)	(.024)	(.085)	(.084)	(.051)
Default (iid) SE(β)	[.009]	[.020]	[.019]	[.021]	[.014]
R^2	.015	.021	.016	.008	.014
σ_α			.181		.161
σ_η	.283		.232		.233
λ	0		1		.585
N	5320	532	5320	4788	5320

Table 1: Comparison of Panel Data Estimators

Comments

- The Within FE estimate of .168 is much higher than POLS estimate of .083

- The FD FE estimate of .109 is also higher than POLS

- RE estimate of .119 is in between Between and Within FE

- POLS default SE of .009 is much smaller than panel robust SE of .030

- Apparent efficiency loss (higher SEs) from using FE estimators

Autocorrelations of Pooled OLS Residuals

	u_{79}	u_{80}	u_{81}	u_{82}	u_{83}	u_{84}	u_{85}	u_{86}	u_{87}	u_{88}
u_{79}	1									
u_{80}	.33	1								
u_{81}	.44	.4	1							
u_{82}	.3	.31	.57	1						
u_{83}	.21	.23	.37	.47	1					
u_{84}	.2	.23	.32	.34	.64	1				
u_{85}	.24	.32	.41	.35	.39	.58	1			
u_{86}	.2	.19	.28	.25	.31	.35	.4	1		
u_{87}	.2	.32	.33	.29	.31	.34	.39	.35	1	
u_{88}	.16	.25	.3	.26	.21	.25	.34	.55	.53	1

- Autocorrelations $Cor(u_{it}, u_{is})$ resemble pattern of equi-correlation RE model

Hausman Specification Test for FE vs. RE

$$y_{it} = \mathbf{x}_{it}'\boldsymbol{\beta} + \alpha_i + \eta_{it}$$

The hypotheses to be tested are

$$\begin{aligned} H_0 &: E[\mathbf{x}_{it}\alpha_i] = 0 \text{ (RE estimation)} \\ H_1 &: E[\mathbf{x}_{it}\alpha_i] \neq 0 \text{ (FE estimation)} \end{aligned}$$

Hausman and Taylor (1981, Ecta) considered a test statistic based on

$$\hat{\mathbf{q}} = \hat{\boldsymbol{\beta}}_{FE} - \hat{\boldsymbol{\beta}}_{RE}$$

in the context of maximum likelihood estimation under the equi-correlation RE framework.

Intuition: Under H_0, both $\hat{\boldsymbol{\beta}}_{FE}$ and $\hat{\boldsymbol{\beta}}_{RE}$ are consistent (but $\hat{\boldsymbol{\beta}}_{RE}$ is efficient) so that

$$\hat{\mathbf{q}} \xrightarrow{p} \mathbf{0}$$

Under H_1, $\hat{\boldsymbol{\beta}}_{RE}$ is not consistent but $\hat{\boldsymbol{\beta}}_{FE}$ is consistent so that

$$\hat{\mathbf{q}} \overset{p}{\nrightarrow} \mathbf{0}$$

Therefore, consider the test statistic

$$H = n\hat{\mathbf{q}} \left(\widehat{\mathsf{avar}}(\hat{\mathbf{q}})\right)^{-1} \hat{\mathbf{q}}$$

If H is big then reject H_0; otherwise do not reject H_0.

Q1: What is $\widehat{\mathsf{avar}}(\hat{\mathbf{q}})$?

Q2: What is the asymptotic distribution of H?

Hausman Test Principle

In general, consider two estimators $\hat{\boldsymbol{\theta}}$ and $\tilde{\boldsymbol{\theta}}$ of $\boldsymbol{\theta} \in \mathbb{R}^L$ such that

$$\begin{aligned} H_0 &: \hat{\boldsymbol{\theta}} - \tilde{\boldsymbol{\theta}} \xrightarrow{p} \mathbf{0} \text{ and } \sqrt{n}\left(\hat{\boldsymbol{\theta}} - \tilde{\boldsymbol{\theta}}\right) \xrightarrow{d} N(0, \mathbf{V}) \\ H_a &: \hat{\boldsymbol{\theta}} - \tilde{\boldsymbol{\theta}} \stackrel{p}{\nrightarrow} \mathbf{0} \end{aligned}$$

The Hausman statistic is

$$H = \left(\hat{\boldsymbol{\theta}} - \tilde{\boldsymbol{\theta}}\right)' \left(n^{-1}\hat{\mathbf{V}}\right)^{-1} \left(\hat{\boldsymbol{\theta}} - \tilde{\boldsymbol{\theta}}\right)$$

and under H_0

$$H \sim \chi^2(L)$$

What is V?

$$n^{-1}V = var(\hat{\boldsymbol{\theta}} - \tilde{\boldsymbol{\theta}}) = var(\hat{\boldsymbol{\theta}}) + var(\tilde{\boldsymbol{\theta}}) - 2cov(\hat{\boldsymbol{\theta}}, \tilde{\boldsymbol{\theta}})$$

Result (Hausman, 1978). If $\hat{\boldsymbol{\theta}}$ is efficient under H_0 then

$$cov(\hat{\boldsymbol{\theta}}, \tilde{\boldsymbol{\theta}}) = var(\hat{\boldsymbol{\theta}})$$

and

$$var(\hat{\boldsymbol{\theta}} - \tilde{\boldsymbol{\theta}}) = var(\tilde{\boldsymbol{\theta}}) - var(\hat{\boldsymbol{\theta}})$$

Hence,

$$H = \left(\hat{\boldsymbol{\theta}} - \tilde{\boldsymbol{\theta}}\right)' \left(\widehat{var}(\tilde{\boldsymbol{\theta}}) - \widehat{var}(\hat{\boldsymbol{\theta}})\right)^{-1} \left(\hat{\boldsymbol{\theta}} - \tilde{\boldsymbol{\theta}}\right)$$

Comments

- Hausman test only requires $\widehat{var}(\tilde{\boldsymbol{\theta}})$ and $\widehat{var}(\hat{\boldsymbol{\theta}})$ and not $cov(\hat{\boldsymbol{\theta}}, \tilde{\boldsymbol{\theta}})$

- It is possible for $\widehat{var}(\tilde{\boldsymbol{\theta}}) - \widehat{var}(\hat{\boldsymbol{\theta}})$ to be negative definite in finite samples

- It is possible for $\widehat{var}(\tilde{\boldsymbol{\theta}}) - \widehat{var}(\hat{\boldsymbol{\theta}})$ to be less than full rank in finite samples

- Hausman test can often be computed using an auxiliary regression

Hausman Test for FE vs. RE

Under the RE model based on the assumptions

$$u_i \sim \text{iid } (0, \sigma_u^2)$$
$$\eta_{it} \sim \text{iid } (0, \sigma_\eta^2)$$

The GLS estimator is asymptotically equivalent to the MLE under normality and is efficient. Hence, for $\hat{\mathbf{q}} = \hat{\boldsymbol{\beta}}_{FE} - \hat{\boldsymbol{\beta}}_{RE}$

$$H = \hat{\mathbf{q}} \left(\widehat{var}(\hat{\boldsymbol{\beta}}_{FE}) - \widehat{var}(\hat{\boldsymbol{\beta}}_{RE}) \right)^{-1} \hat{\mathbf{q}} \sim \chi^2(L)$$

Comment:

- The H statistic is not robust to serial correlation and heteroskedasticity

Hausman Statistic Based on Auxiliary Regression

An asymptotically equivalent version of the Hausman test can be computed as the Wald test of $\boldsymbol{\gamma} = \mathbf{0}$ in the auxiliary regression

$$
\begin{aligned}
y_{it} - \hat{\lambda}\bar{y}_i &= (\mathbf{w}_{it} - \hat{\lambda}\bar{\mathbf{w}}_i)'\boldsymbol{\delta} + \boldsymbol{\gamma}'(\mathbf{x}_{it} - \bar{\mathbf{x}}_i) + v_{it} \\
\mathbf{w}_{it} &= (1, \mathbf{x}'_{it})' \\
\hat{\lambda} &= 1 - \frac{\hat{\sigma}^2_\eta}{\sqrt{T\hat{\sigma}^2_u + \hat{\sigma}^2_\eta}}
\end{aligned}
$$

Why? Because the RE estimator is obtained when $\boldsymbol{\gamma} = \mathbf{0}$ is imposed on the auxiliary regression.

Comment

- Auxiliary regression approach avoids problems of $\hat{\mathbf{q}}$ having non-full rank variance or negative definite variance.

- Wald test of $\boldsymbol{\gamma} = \mathbf{0}$ can be made robust to serial correlation and heteroskedasticity using panel robust covariance estimates

Testing FE vs. RE in Hours and Wages Panel Regression

$$\ln hrs_{it} = \beta \ln wages_{it} + \alpha_i + \eta_{it}$$

Using the original form of the Hausman test, we have

$$H = \frac{(\hat{\beta}_{FE} - \hat{\beta}_{RE})^2}{\widehat{var}(\hat{\beta}_{FE}) - \widehat{var}(\hat{\beta}_{RE})} = \frac{(.168 - .119)^2}{(.019^2 - .014^2)} = 14 > \chi^2_{.95}(1) = 3.84$$

so we reject $H_0 : E[\ln wages_{it} \times \alpha_i] = 0$ at the 5% level.

Comment:

- Test may not be reliable because of serial correlation and heteroskedasticity

The auxiliary regression approach based on the regression

$$\begin{aligned} y_{it} - \hat{\lambda}\bar{y}_i &= (\mathbf{w}_{it} - \hat{\lambda}\bar{\mathbf{w}}_i)'\boldsymbol{\delta} + \gamma(x_{it} - \bar{x}_i) + v_{it} \\ y_{it} &= \ln hrs_{it},\ \mathbf{w}_{it} = (1, \ln wages_{it})',\ x_{it} = \ln wages_{it} \end{aligned}$$

with panel robust standard errors gives the statistic

$$t^2_{\gamma=0} = (1.28)^2 = 1.65 < \chi^2_{.95}(1) = 3.84$$

so we cannot reject $H_0 : E[\ln wages_{it} \times \alpha_i] = 0$ at the 5% level.