Econ 582
Cointegration

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Cointegration

- The time series models discussed so far are appropriate for modeling $I(0)$ data, like asset returns or growth rates of macroeconomic time series.
- Economic theory, however, often implies equilibrium relationships between the levels of time series variables that are best described as being $I(1)$.
- Similarly, arbitrage arguments imply that the $I(1)$ prices of certain financial time series are linked.
- The statistical concept of cointegration is required to make sense of regression models and VAR models with $I(1)$ data.

Spurious Regression

If some or all of the variables in a regression are $I(1)$ then the usual statistical results may or may not hold. One important case in which the usual statistical results do not hold is *spurious regression*, when all the regressors are $I(1)$ and not cointegrated. That is, there is no linear combination of the variables that is $I(0)$.

Example 1 *spurious regression using simulated data*

Consider two independent and not cointegrated $I(1)$ processes y_{1t} and y_{2t}

$$\begin{aligned} y_{it} &= y_{it-1} + \varepsilon_{it},\ t = 1, \ldots, 250 \\ \varepsilon_{it} &\sim GWN(0,1),\ i = 1,2 \end{aligned}$$

Estimated levels and differences regressions

$$\begin{aligned} y_1 &= \underset{(0.39)}{6.74} + \underset{(0.05)}{0.40} \cdot y_2,\ R^2 = 0.21 \\ \Delta y_1 &= \underset{(0.07)}{-0.06} + \underset{(0.06)}{0.03} \cdot \Delta y_2,\ R^2 = 0.00 \end{aligned}$$

Statistical Implications of Spurious Regression

Let $\mathbf{Y}_t = (y_{1t}, \ldots, y_{nt})'$ denote an $(n \times 1)$ vector of $I(1)$ time series that are *not cointegrated*. Write

$$\mathbf{Y}_t = (y_{1t}, \mathbf{Y}_{2t}')',$$

and consider regressing of y_{1t} on $\mathbf{Y}_{2t}$ giving

$$y_{1t} = \hat{\boldsymbol{\beta}}_2' \mathbf{Y}_{2t} + \hat{u}_t$$

Since y_{1t} is not cointegrated with $\mathbf{Y}_{2t}$

- true value of $\boldsymbol{\beta}_2$ is zero

- The above is a *spurious regression* and $\hat{u}_t \sim I(1)$.

The following results about the behavior of $\hat{\boldsymbol{\beta}}_2$ in the spurious regression are due to Phillips (1986):

- $\hat{\boldsymbol{\beta}}_2$ does not converge in probability to zero but instead converges in distribution to a non-normal random variable not necessarily centered at zero. This is the spurious regression phenomenon.

- The usual OLS t-statistics for testing that the elements of $\boldsymbol{\beta}_2$ are zero diverge to $\pm\infty$ as $T \to \infty$. Hence, with a large enough sample it will appear that $\mathbf{Y}_t$ is cointegrated when it is not if the usual asymptotic normal inference is used.

- The usual R^2 from the regression converges to unity as $T \to \infty$ so that the model will appear to fit well even though it is misspecified.

Intuition

Recall, with I(1) data sample moments converge to functions of Brownian motion. Consider two independent and not cointegrated I(1) processes y_{1t} and y_{2t} :

$$y_{it} = y_{it-1} + \varepsilon_{it}, \varepsilon_{it} \sim WN(0, \sigma_i^2),\ i = 1, 2$$

Then it can be shown that

$$T^{-2} \sum_{t=1}^{T} y_{it} \xrightarrow{d} \sigma_i^2 \int_0^1 W_i(r)^2 dr,\ i = 1, 2$$
$$T^{-2} \sum_{t=1}^{T} y_{1t} y_{2t} \xrightarrow{d} \sigma_1 \sigma_2 \int_0^1 W_1(r) W_2(r) dr$$

where $W_1(r)$ and $W_2(r)$ are independent Wiener processes.

In the regression

$$y_{1t} = \hat{\beta} y_{2t} + \hat{u}_t$$

Phillips showed that

$$\begin{aligned}
\hat{\beta} &= \left(T^{-2} \sum_{t=1}^{T} y_{2t} \right)^{-1} T^{-2} \sum_{t=1}^{T} y_{1t} y_{2t} \\
&\xrightarrow{d} \left(\sigma_i^2 \int_0^1 W_i(r)^2 dr \right)^{-1} \sigma_1 \sigma_2 \int_0^1 W_1(r) W_2(r) dr
\end{aligned}$$

and so

$$\hat{\beta} \overset{p}{\nrightarrow} 0 \text{ but } \hat{\beta} \xrightarrow{d} \text{ random variable!}$$

Cointegration

Let $\mathbf{Y}_t = (y_{1t}, \ldots, y_{nt})'$ denote an $(n \times 1)$ vector of $I(1)$ time series. $\mathbf{Y}_t$ is *cointegrated* if there exists an $(n \times 1)$ vector $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)'$ such that

$$\boldsymbol{\beta}'\mathbf{Y}_t = \beta_1 y_{1t} + \cdots + \beta_n y_{nt} \sim I(0)$$

In words, the nonstationary time series in $\mathbf{Y}_t$ are cointegrated if there is a linear combination of them that is stationary or $I(0)$.

- The linear combination $\boldsymbol{\beta}'\mathbf{Y}_t$ is often motivated by economic theory and referred to as a *long-run equilibrium* relationship.

- Intuition: $I(1)$ time series with a long-run equilibrium relationship cannot drift too far apart from the equilibrium because economic forces will act to restore the equilibrium relationship.

Normalization

The cointegration vector $\boldsymbol{\beta}$ is not unique since for any scalar c

$$c\boldsymbol{\beta}'\mathbf{Y}_t = \boldsymbol{\beta}^{*\prime}\mathbf{Y}_t \sim I(0)$$

Hence, some *normalization* assumption is required to uniquely identify $\boldsymbol{\beta}$. A typical normalization is

$$\boldsymbol{\beta} = (1, -\beta_2, \ldots, -\beta_n)'$$

so that

$$\boldsymbol{\beta}'\mathbf{Y}_t = y_{1t} - \beta_2 y_{2t} - \cdots - \beta_n y_{nt} \sim I(0)$$

or

$$\begin{aligned} y_{1t} &= \beta_2 y_{2t} + \cdots + \beta_n y_{nt} + u_t \\ u_t &\sim I(0) = \text{cointegrating residual} \end{aligned}$$

Multiple Cointegrating Relationships

If the $(n \times 1)$ vector $\mathbf{Y}_t$ is cointegrated there may be $0 < r < n$ linearly independent cointegrating vectors. For example, let $n = 3$ and suppose there are $r = 2$ cointegrating vectors

$$\begin{aligned} \boldsymbol{\beta}_1 &= (\beta_{11}, \beta_{12}, \beta_{13})' \\ \boldsymbol{\beta}_2 &= (\beta_{21}, \beta_{22}, \beta_{23})' \end{aligned}$$

Then

$$\begin{aligned} \boldsymbol{\beta}_1'\mathbf{Y}_t &= \beta_{11}y_{1t} + \beta_{12}y_{2t} + \beta_{13}y_{3t} \sim I(0) \\ \boldsymbol{\beta}_2'\mathbf{Y}_t &= \beta_{21}y_{1t} + \beta_{22}y_{2t} + \beta_{23}y_{3t} \sim I(0) \end{aligned}$$

The (3×2) matrix

$$\mathbf{B}' = \begin{pmatrix} \boldsymbol{\beta}_1' \\ \boldsymbol{\beta}_2' \end{pmatrix} = \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{33} \end{pmatrix}$$

forms a *basis* for the space of cointegrating vectors.

The linearly independent vectors $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ in the cointegrating basis $\mathbf{B}$ are not unique unless some normalization assumptions are made. Furthermore, any linear combination of $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$, e.g. $\boldsymbol{\beta}_3 = c_1\boldsymbol{\beta}_1 + c_2\boldsymbol{\beta}_2$ where c_1 and c_2 are constants, is also a cointegrating vector.

Examples of Cointegration and Common Trends in Economics and Finance

Cointegration naturally arises in economics and finance. In economics, cointegration is most often associated with economic theories that imply equilibrium relationships between time series variables:

- The permanent income model implies cointegration between consumption and income, with consumption being the common trend.
- Money demand models imply cointegration between money, income, prices and interest rates.

- Growth theory models imply cointegration between income, consumption and investment, with productivity being the common trend.
- Purchasing power parity implies cointegration between the nominal exchange rate and foreign and domestic prices.
- Covered interest rate parity implies cointegration between forward and spot exchange rates.
- The Fisher equation implies cointegration between nominal interest rates and inflation.
- The expectations hypothesis of the term structure implies cointegration between nominal interest rates at different maturities.

- The present value model of stock prices states that a stock's price is an expected discounted present value of its expected future dividends or earnings.

Remarks:

- The equilibrium relationships implied by these economic theories are referred to as *long-run equilibrium* relationships, because the economic forces that act in response to deviations from equilibriium may take a long time to restore equilibrium. As a result, cointegration is modeled using long spans of low frequency time series data measured monthly, quarterly or annually.

- In finance, cointegration may be a high frequency relationship or a low frequency relationship. Cointegration at a high frequency is motivated by arbitrage arguments.

- The *Law of One Price* implies that identical assets must sell for the same price to avoid arbitrage opportunities. This implies cointegration between the prices of the same asset trading on different markets, for example.
- Similar arbitrage arguments imply cointegration between spot and futures prices, and spot and forward prices, and bid and ask prices.

Here the terminology long-run equilibrium relationship is somewhat misleading because the economic forces acting to eliminate arbitrage opportunities work very quickly. Cointegration is appropriately modeled using short spans of high frequency data in seconds, minutes, hours or days.

Cointegration and Common Trends

If the $(n \times 1)$ vector time series $\mathbf{Y}_t$ is cointegrated with $0 < r < n$ cointegrating vectors then there are $n - r$ *common* $I(1)$ *stochastic trends.*

For example, let

$$\mathbf{Y}_t = (y_{1t}, y_{2t})' \sim I(1),\ \boldsymbol{\varepsilon}_t = (\varepsilon_{1t}, \varepsilon_{2t}, \varepsilon_{3t})' \sim I(0)$$

and suppose that $\mathbf{Y}_t$ is cointegrated with cointegrating vector $\boldsymbol{\beta} = (1, -\beta_2)'$. This cointegration relationship may be represented as

$$\begin{aligned} y_{1t} &= \beta_2 \sum_{s=1}^{t} \varepsilon_{1s} + \varepsilon_{3t} \\ y_{2t} &= \sum_{s=1}^{t} \varepsilon_{1s} + \varepsilon_{2t} \end{aligned}$$

The common stochastic trend is $\sum_{s=1}^{t} \varepsilon_{1s}$.

Remark:

The cointegrating vector $\boldsymbol{\beta} = (1, -\beta_2)'$ annihilates the common stochastic trend:

$$
\begin{aligned}
\boldsymbol{\beta}'\mathbf{Y}_t &= (1, -\beta_2)' \begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = y_{1t} - \beta_2 y_{2t} \\
&= \beta_2 \sum_{s=1}^{t} \varepsilon_{1s} + \varepsilon_{3t} - \beta_2 \left(\sum_{s=1}^{t} \varepsilon_{1s} + \varepsilon_{2t} \right) \\
&= \varepsilon_{3t} - \beta_2 \varepsilon_{2t} \sim I(0).
\end{aligned}
$$

Some Simulated Cointegrated Systems

Cointegrated systems may be conveniently simulated using Phillips' (1991) *triangular representation*. For example, consider a bivariate cointegrated system for $\mathbf{Y}_t = (y_{1t}, y_{2t})'$ with cointegrating vector $\boldsymbol{\beta} = (1, -\beta_2)'$. A triangular representation has the form

$$
\begin{aligned}
y_{1t} &= \beta_2 y_{2t} + u_t, \text{ where } u_t \sim I(0) \\
y_{2t} &= y_{2t-1} + v_t, \text{ where } v_t \sim I(0)
\end{aligned}
$$

- The first equation describes the long-run equilibrium relationship with an $I(0)$ disequilibrium error u_t.

- The second equation specifies y_{2t} as the common stochastic trend with innovation v_t:

$$y_{2t} = y_{20} + \sum_{j=1}^{t} v_j.$$

- In general, the innovations u_t and v_t may be contemporaneously and serially correlated. The time series structure of these innovations characterizes the short-run dynamics of the cointegrated system.

Example: Bivariate system with $\boldsymbol{\beta} = (1, -1)'$

$$\begin{aligned} y_{1t} &= y_{2t} + u_t \\ y_{2t} &= y_{2t-1} + v_t \\ u_t &= 0.75 u_{t-1} + \varepsilon_t, \\ \varepsilon_t &\sim iid\ N(0, (0.5)^2), \\ v_t &\sim iid\ N(0, (0.5)^2) \end{aligned}$$

Remark:

The bivariate system with $\boldsymbol{\beta} = (1, -1)'$ can be used to model the behavior of the logarithm of spot and forward prices, spot and futures prices, stock prices and dividends, or consumption and income.

Trivariate cointegrated system with 1 cointegrating vector

Let $\boldsymbol{\beta} = (1, -\beta_1, -\beta_2)'$. Then the triangular system can be written as

$$\begin{aligned} y_{1t} &= \beta_1 y_{2t} + \beta_2 y_{3t} + u_t,\ u_t \sim I(0) \\ y_{2t} &= y_{2t-1} + v_t,\ v_t \sim I(0) \\ y_{3t} &= y_{3t-1} + w_t,\ w_t \sim I(0) \end{aligned}$$

An example of a trivariate cointegrated system with one cointegrating vector is a system of nominal exchange rates, home country price indices and foreign country price indices (all in logs). A cointegrating vector $\boldsymbol{\beta} = (1, -1, 1)'$ implies that the real exchange rate is stationary.

Example: Simulated trivariate cointegrated system with $\boldsymbol{\beta} = (1, -0.5, -0.5)'$

$$\begin{aligned} y_{1t} &= 0.5 y_{2t} + 0.5 y_{3t} + u_t, \\ u_t &= 0.75 u_{t-1} + \varepsilon_t, \varepsilon_t \sim iid\ N(0, (0.5)^2) \\ y_{2t} &= y_{2t-1} + v_t,\ v_t \sim iid\ N(0, (0.5)^2) \\ y_{3t} &= y_{3t-1} + w_t,\ w_t \sim iid\ N(0, (0.5)^2) \end{aligned}$$

Trivariate cointegrated system with 2 cointegrating vectors

A triangular representation for this system with cointegrating vectors $\boldsymbol{\beta}_1 = (1, 0, -\beta_{13})'$ and $\boldsymbol{\beta}_2 = (0, 1, -\beta_{23})'$ is

$$\begin{aligned} y_{1t} &= \beta_{13} y_{3t} + u_t, \;\; u_t \sim I(0) \\ y_{2t} &= \beta_{23} y_{3t} + v_t, \;\; v_t \sim I(0) \\ y_{3t} &= y_{3t-1} + w_t, \;\; w_t \sim I(0) \end{aligned}$$

An example in finance of such a system is the term structure of interest rates where y_3 represents the short rate and y_1 and y_2 represent two different long rates. The cointegrating relationships would indicate that the spreads between the long and short rates are stationary.

Example: Trivariate system with $\boldsymbol{\beta}_1 = (1, 0, -1)'$, $\beta_2 = (0, 1, -1)'$

$$\begin{aligned} y_{1t} &= y_{3t} + u_t, \\ u_t &= 0.75 u_{t-1} + \varepsilon_t, \varepsilon_t \sim iid\ N(0, (0.5)^2) \\ y_{2t} &= y_{3t} + v_t, \\ v_t &= 0.75 v_{t-1} + \eta_t, \eta_t \sim iid\ N(0, (0.5)^2) \\ y_{3t} &= y_{3t-1} + w_t, \;\; w_t \sim iid\ N(0, (0.5)^2) \end{aligned}$$

Cointegration and Error Correction Models

Consider a bivariate $I(1)$ vector $\mathbf{Y}_t = (y_{1t}, y_{2t})'$ and assume that $\mathbf{Y}_t$ is cointegrated with cointegrating vector $\boldsymbol{\beta} = (1, -\beta_2)'$ so that $\boldsymbol{\beta}'\mathbf{Y}_t = y_{1t} - \beta_2 y_{2t}$ is $I(0)$. Engle and Granger's famous (1987) *Econometrica* paper showed that cointegration implies the existence of an *error correction model* (ECM) of the form

$$\begin{aligned}
\Delta y_{1t} &= c_1 + \alpha_1(y_{1t-1} - \beta_2 y_{2t-1}) \\
&\quad + \sum_j \psi_{11}^j \Delta y_{1t-j} + \sum_j \psi_{12}^j \Delta y_{2t-j} + \varepsilon_{1t} \\
\Delta y_{2t} &= c_2 + \alpha_2(y_{1t-1} - \beta_2 y_{2t-1}) \\
&\quad + \sum_j \psi_{21}^j \Delta y_{1t-j} + \sum_j \psi_{22}^j \Delta y_{2t-j} + \varepsilon_{2t}
\end{aligned}$$

The ECM links the long-run equilibrium relationship implied by cointegration with the short-run dynamic adjustment mechanism that describes how the variables react when they move out of long-run equilibrium.

Example 2 *Bivariate ECM for consumption and income*

Let y_t denote the log of real income and c_t denote the log of consumption and assume that $\mathbf{Y}_t = (y_t, c_t)'$ is $I(1)$. The Permanent Income Hypothesis implies that income and consumption are cointegrated with $\boldsymbol{\beta} = (1, -1)'$:

$$\begin{aligned}
c_t &= \mu + y_t + u_t \\
\mu &= E[c_t - y_t] = \text{ mean savings rate} \\
&\quad u_t \sim I(0)
\end{aligned}$$

Suppose the ECM has the form

$$\begin{aligned}
\Delta y_t &= c_y + \alpha_y(c_{t-1} - y_{t-1} - \mu) + \varepsilon_{yt} \\
\Delta c_t &= c_c + \alpha_c(c_{t-1} - y_{t-1} - \mu) + \varepsilon_{ct}
\end{aligned}$$

The first equation relates the growth rate of income to the lagged disequilibrium error $c_{t-1} - y_{t-1} - \mu$, and the second equation relates the growth rate of consumption to the lagged disequilibrium as well. The reactions of y_t and c_t to the disequilibrium error are captured by the *adjustment coefficients* α_y and α_c.

Consider the special case

$$\begin{aligned} \Delta y_t &= c_y + 0.5(c_{t-1} - y_{t-1} - \mu) + \varepsilon_{yt}, \\ \Delta c_t &= c_c + \varepsilon_{ct}. \end{aligned}$$

Consider three situations:

1. $c_{t-1} - y_{t-1} - \mu = 0$. Then

$$\begin{aligned} E[\Delta y_t | \mathbf{Y}_{t-1}] &= c_y = \text{ long-run growth rate} \\ E[\Delta c_t | \mathbf{Y}_{t-1}] &= c_c = \text{ long-run growth rate} \end{aligned}$$

2. $c_{t-1} - y_{t-1} - \mu > 0$. Then

$$E[\Delta y_t | \mathbf{Y}_{t-1}] = c_y + 0.5(c_{t-1} - y_{t-1} - \mu) > c_y$$

Here the consumption has increased above its long-run mean (positive disequilibrium error) and the ECM predicts that y_t will grow faster than its long-run rate c_y to restore the consumption-income ratio its long-run mean.

3. $c_{t-1} - y_{t-1} - \mu < 0$. Then

$$E[\Delta y_t | \mathbf{Y}_{t-1}] = c_y + 0.5(c_{t-1} - y_{t-1} - \mu) < c_y$$

Here consumption-income ratio has decreased below its long-run mean (negative disequilibrium error) and the ECM predicts that y_t will grow more slowly than its long-run rate to restore the consumption-income ratio to its long-run mean.