Dynamic Regression Models

I. Distributed lag (DL) models

Ex. $$y_t = \alpha + \beta_0 x_t + \beta_1 x_{t-1} + \epsilon_t$$

x_t is strictly exogenous (e.g. independent of y_t)

Could be dynamic demand function

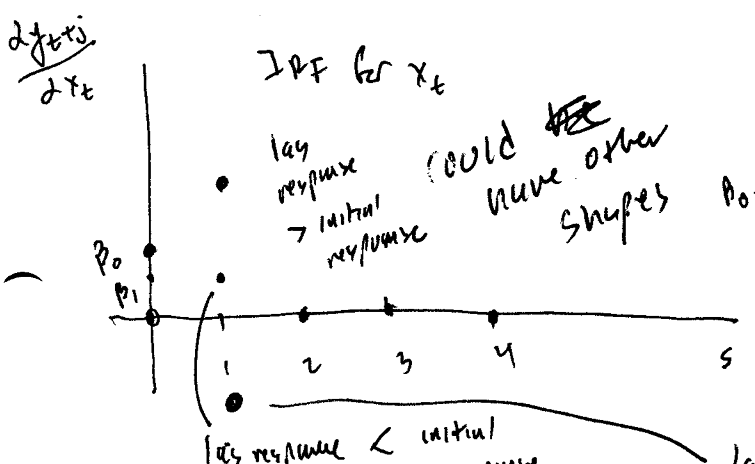
$y_t = \ln$ quantity

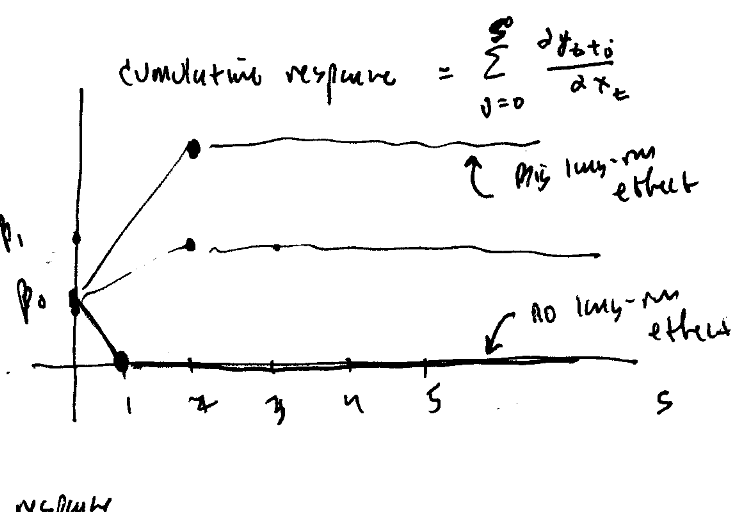
$x_t = \ln$ price

$$\frac{\partial y_t}{\partial x_t} = \beta_0 = \text{contemporaneous (price elasticity) multiplier}$$

$$\frac{\partial y_t}{\partial x_{t-1}} = \frac{\partial y_{t+1}}{\partial x_t} = \beta_1 = \text{lag 1 multiplier or lag response}$$

$$\frac{\partial y_t}{\partial x_{t-j}} = \frac{\partial y_{t+j}}{\partial x_t} = 0 \qquad j > 1$$





lag response

Cumulative impact of a change in x_t $\overset{def}{=}$ long-run effect

$$\lim_{s \to \infty} \sum_{j=0}^{s} \frac{\partial y_{t+s}}{\partial x_t} = \frac{\partial y_t}{\partial x_t} + \frac{\partial y_{t+1}}{\partial x_t} + \cdots$$

In the example we have

$$\text{long-run effect} = \frac{\partial y_t}{\partial x_t} + \frac{\partial y_{t+1}}{\partial x_t} = \beta_0 + \beta_1$$

= sum of lag weights.

<u>Alternative derivation</u> (long-run static soln)

Suppose $x_t = x_{t-1} = x^*$ and $\epsilon_t = 0$. Then define the long-run static soln for y_t as

$$y^* = \alpha + \beta_0 x^* + \beta_1 x^*$$

$$= \alpha + (\beta_0 + \beta_1) x^*$$

<u>Note</u>: $\frac{\partial y^*}{\partial x^*} = \beta_0 + \beta_1 =$ long-run impact.

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Lag operator Notation

$$y_t = \alpha + \beta_0 x_t + \beta_1 L x_t + \epsilon_t$$

$$= \alpha + (\beta_0 + \beta_1 L)\, x_t + \epsilon_t$$

$$= \alpha + \beta(L)\, x_t + \epsilon_t$$

where

$$\beta(L) = \beta_0 + \beta_1 L$$

Then

$$\frac{\partial y^*}{\partial x^*} = \beta_0 + \beta_1 = \beta(1) = \beta(L)\Big|_{L=1}$$

General DL model

$$y_t = \alpha + \beta_0 x_t + \beta_1 x_{t-1} + \cdots + \beta_p x_{t-p} + \epsilon_t$$

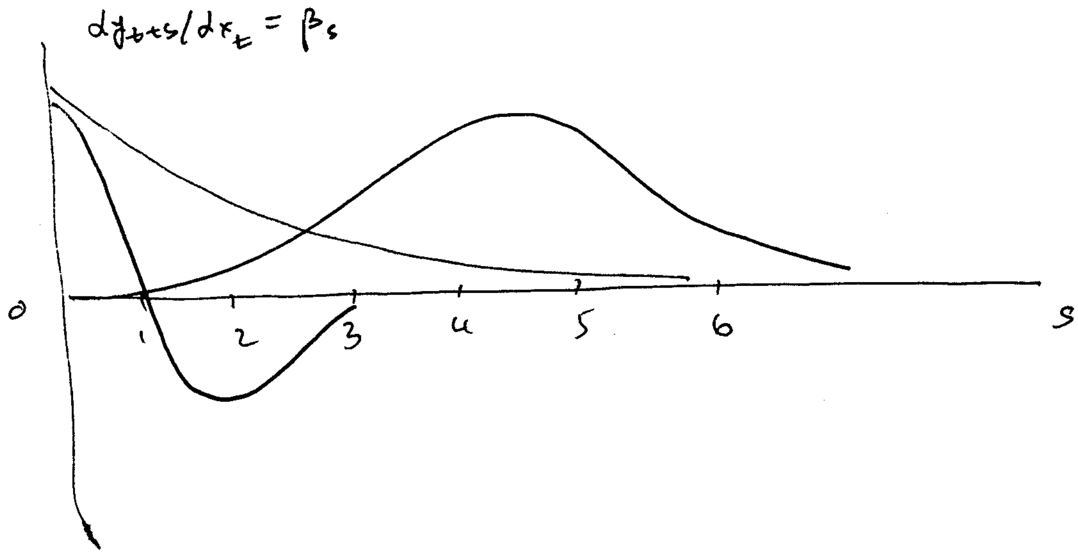
x_t is exogenous

$$\frac{\partial y_t}{\partial x_{t-j}} = \frac{\partial y_{t+j}}{\partial x_t} = \beta_j \qquad j \le p$$

In lag notation

$$y_t = \alpha + \beta(L) x_t + \epsilon_t, \quad \beta(L) = \beta_0 + \beta_1 L + \cdots + \beta_p L^p$$

General DL models can have very rich lag responses.
For example



Unrestricted Estimation of DL models

1. OLS is just fine provided x_t is exogenous

2. How to choose lag lengths?

   - General-to-specific strategy

(1) Start with max lag q_{max} & estimate DL

$$y_t = \alpha + \beta_0 x_t + \cdots + \beta_{q_{max}} x_{t-q_{max}} + \epsilon_t$$

(2) Test $H_0: \beta_{q_{max}} = 0$ vs. $H_1: \beta_{q_{max}} \neq 0$ at 5% level say.

(or $H_0: \beta_{q_{max}} = \beta_{q_{max}-1} = 0$ vs. $H_1:$)

· If ~~don't~~ reject H_0, stop and set $q = q_{max}$

If ~~reject~~ do not reject H_0 then set $q_{max} = q_{max} - 1$

and repeat steps (1)-(2) ~~step taken~~

- Use model selection criteria like AIC (Akaike information criterion)

$$AIC(q) = \ln \hat{\sigma}^2 + T^{-1} 2q$$

where $\hat{\sigma}^2$ is residual variance from DL model with q lags.

Best model satisfies

$$\min_{q < q_{max}} AIC(q)$$

where all model are fit with the same sample size T.

<u>Intuition:</u> $\hat{\sigma}^2$ small $\Rightarrow$ good fit

$T^{-1} 2 \cdot q$ = penalty term for extra parameters (like $\bar{R}^2$)

3. $x_t, x_{t-1}, x_{t-2}, \ldots$ may be highly correlated particularly if using levels of trending data

$\Rightarrow$ individual $\hat{\beta}$'s not significant
jointly $\hat{\beta}$s are significant

Remark: In some cases, collinearity among variables can be mitigated by reparameterization

Eg. $y_t = \alpha + \beta_0 x_t + \beta_1 x_{t-1} + \epsilon_t$

Problem: x_t, x_{t-1} highly correlated.

Soln: rewrite model as

$$y_t = \alpha + \beta_0 x_t - \beta_0 x_{t-1} + \beta_0 x_{t-1} + \beta_1 x_{t-1} + \epsilon_t$$

$$= \alpha + \beta_0 \Delta x_t + (\beta_0 + \beta_1) x_{t-1} + \epsilon_t$$

or

$$y_t = \alpha + \beta_0 \Delta x_t + \gamma x_{t-1} + \epsilon_t$$

Here, Δx_t & x_{t-1} will be much less correlated than x_t & x_{t-1}. Note: $\beta_1 = \gamma - \beta_0$

Also, OLS on both models give identical fits since RHS of new model is a linear transformation of old model

Restricting the lag structure: The Geometric lag model

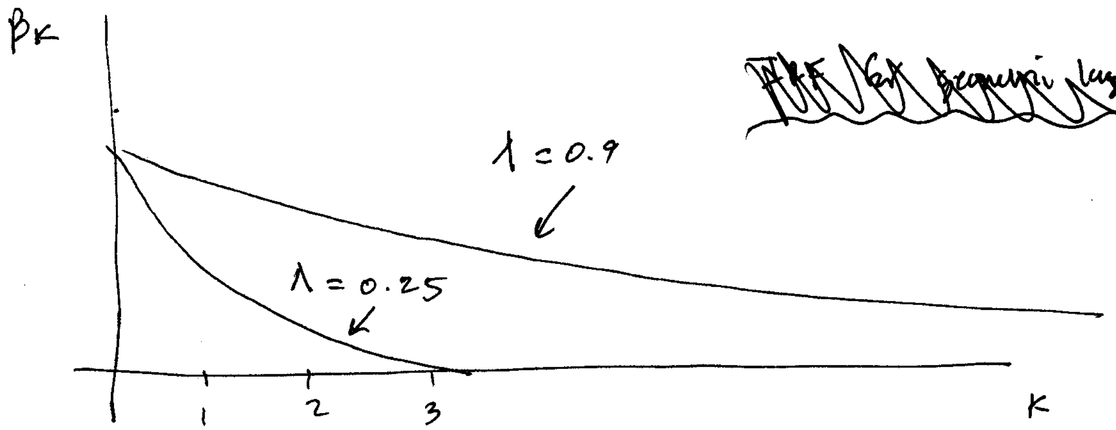
Unrestricted model : $y_t = \alpha + \sum_{k=0}^{\infty} \beta_k x_{t-k} + \epsilon_t$

DL(∞)

Assume: $\beta_k = \beta(1-\lambda)\lambda^k \qquad 0 \le \lambda < 1$

λ measures persistence of lags

- $\lambda \approx 0 \Rightarrow$ low persistence
- $\lambda \approx 1 \Rightarrow$ large persistence



Note: Since $\lambda < 1 \Rightarrow \lim_{k \to \infty} \beta_k = 0$.

Substituting $\beta_k = \beta(1-\lambda)\lambda^k$ into DL(∞) gives

$$y_t = \alpha + \beta \sum_{k=0}^{\infty} (1-\lambda)\lambda^k x_{t-k} + \epsilon_t$$

$$= \alpha + \beta \cdot B(L)\, x_t + \epsilon_t$$

where

$$B(L) = (1-\lambda) \sum_{k=0}^{\infty} \lambda^k L^k$$

$$= (1-\lambda)(1-\lambda L)^{-1} \qquad \text{since } |\lambda| < 1.$$

Hence

$$y_t = \alpha + \beta(1-\lambda)(1-\lambda L)^{-1} x_t + \epsilon_t$$

<u>IRF for geometric lag</u>

$$\frac{\partial y_t}{\partial x_t} = \beta(1-\lambda)$$

$$\frac{\partial y_{t+s}}{\partial x_t} = \beta(1-\lambda)\lambda^s$$

$$\text{long-run response} = \beta \cdot B(1) = \beta(1-\lambda)(1-\lambda)^{-1}$$

$$= \beta$$

i.e. the long-run "equilibrium" is

$$y^* = \alpha + \beta \cdot x^*$$

which can also be derived via the steady state where $x_t = x^* \ \forall t$ and $\epsilon_t = 0$.