Econ 582 Midterm Exam

This is an open book exam. You must do your own work and you cannot discuss these questions with anyone. If you have questions about any of the problems or believe there is a typo in one of the questions please contact Clarisse or myself immediately. The exam is due on Friday 4/28/00 at the beginning of section.

1. Working with Asymptotic Distributions

Consider the classical regression model

$$\mathbf{y} = \alpha\mathbf{x} + \beta\mathbf{z} + \mathbf{u}$$

where α and β are unknown scalar parameters, $\mathbf{x}$ and $\mathbf{z}$ are $T-$ component vectors of known constants, and $\mathbf{u}$ is a $T-$ component random vector of unobserverabes with mean zero and unit variance. Suppose we are given an initial estimator $\tilde{\beta}$ that is independent of $\mathbf{u}$ and satisfies

$$\sqrt{T}(\tilde{\beta} - \beta) \xrightarrow{d} N(0, 1).$$

Now consider the estimation of α by least squares on the simple regression

$$\mathbf{y} - \tilde{\beta}\mathbf{z} = \alpha\mathbf{x} + \bar{\mathbf{u}},$$

giving the OLS estimate

$$\tilde{\alpha} = (\mathbf{x}'\mathbf{x})^{-1}\mathbf{x}'(\mathbf{y} - \tilde{\beta}\mathbf{z}) = \frac{\mathbf{x}'(\mathbf{y} - \tilde{\beta}\mathbf{z})}{\mathbf{x}'\mathbf{x}}.$$

1. Using the appropriate law of large numbers (LLN), determine $\underset{T\to\infty}{plim}\ \tilde{\alpha}$.

2. Using the appropriate central limit theorems (CLTs), determine the asymptotic distribution of $\tilde{\alpha}$.

2. Testing Nonlinear Hypotheses

Consider the linear regression with two fixed regressors

$$\begin{aligned} y_t &= \beta_0 + \beta_1 x_{1t} + \beta_2 x_{2t} + \varepsilon_t,\ t = 1, \ldots, T \\ \varepsilon_t &\sim iid\ N(0, \sigma^2),\ \boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2)' \end{aligned}$$

and suppose you want to test the non-linear restriction $\beta_1 \beta_2 = 1$. Two algebraically equivalent formulations of the null hypothesis are:

$$\begin{aligned} H_0^1 &: g^1(\boldsymbol{\beta}) = \beta_1 - 1/\beta_2 = 0, \\ H_0^2 &: g^2(\boldsymbol{\beta}) = \beta_1 \beta_2 - 1 = 0. \end{aligned}$$

1. Derive the Wald statistics for testing H_0^1 and H_0^2 . What is the decision rule to reject the null using a 5% (asymptotic) test?

2. Suppose

$$\begin{aligned} \widehat{\boldsymbol{\beta}}_{OLS} &= (0, 100, 0.1)', \\ \widehat{V}ar(\widehat{\boldsymbol{\beta}}_{OLS}) &= \widehat{\sigma}^2 (\mathbf{X}'\mathbf{X})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 0.01 \end{pmatrix}. \end{aligned}$$

   Compute the two Wald statistics and use them to test the nonlinear restriction $\beta_1 \beta_2 = 1$ with a 5% significance level. Do both Wald statistics lead to the same conclusion?

3. Outline how you would compute the likelihood ratio statistic for testing the nonlinear restriction $\beta_1 \beta_2 = 1$. That is, describe the steps required to compute the statistic as if you were writing a computer program in Matlab to do the calculations.

3. MLE and Nonlinear Regression

Consider the nonlinear regression model with fixed regressors and normal errors:

$$\begin{aligned} y_i &= f(\beta, \mathbf{x}_i) + \varepsilon_i,\ i = 1, \ldots, N \\ \varepsilon_i &\sim iid\ N(0, \sigma^2), \\ \boldsymbol{\theta} &= (\beta, \sigma^2)'. \end{aligned}$$

For simplicity, assume that β is a scalar and $\mathbf{x}_i$ is a $k \times 1$ vector. Assume that the function $f(\cdot)$ is continuous and two times differentiable with respect to β. For example, we could have $f(\beta, \mathbf{x}_i) = \beta x_{1i} + \beta^2 x_{2i} + \varepsilon_i$. *Do not* use this function form in the analysis below; work with the general function $f(\beta, x_i)$.

1. What are the nonlinear least squares estimators for β and σ^2? That is, describe the minimization problem you need to solve and an algorithm for how you would compute the nonlinear least squares estimates of β and σ^2.

2. Write out the sample log-likelihood function for $\boldsymbol{\theta}$, $\ln L(\boldsymbol{\theta}; \mathbf{x})$, and compute the score functions for β and σ^2.

3. Concentrate the log-likelihood function with respect to σ^2. That is, solve the first order condition for σ^2 in terms of β and substitute the resulting expression into $\ln L(\boldsymbol{\theta}; \mathbf{x})$. This will give you a function in terms of β and is called the *concentrated log-likelihood* function of β. Denote this function $\ln L^c(\beta; \mathbf{x})$.

4. Show that Maximizing $\ln L^c(\beta; \mathbf{x})$ with respect to β is equivalent to solving for the nonlinear least squares estimator of β. What is the mle for σ^2?

5. How would you estimate the asymptotic variance of the mle for $\boldsymbol{\theta} = (\beta, \sigma^2)'$?

4. Binary Choice Models

Consider a binary choice model where the dependent variable $y_i = 1$ if, say, an individual belongs to a union and $y_i = 0$ otherwise. It is assumed that the probability of joining a union depends on observable characteristics in the $k-$dimensional vector $\mathbf{x}_i$ in the following way:

$$\begin{aligned} \Pr(y_i &= 1|\mathbf{x}_i) = F(\mathbf{x}_i'\boldsymbol{\beta}), \\ \Pr((y_i &= 0|\mathbf{x}_i) = 1 - F(\mathbf{x}_i'\boldsymbol{\beta}), \end{aligned}$$

where $\boldsymbol{\beta}$ is a vector of parameters and $F(\cdot)$ is some function to be specified. Notice that in this formulation $E[y_i|\mathbf{x}_i] = \Pr(y_i = 1|\mathbf{x}_i) = F(\mathbf{x}_i'\boldsymbol{\beta})$. The regression model formulation of the binary choice model is then given by

$$\begin{aligned} y_i &= E[y_i|\mathbf{x}_i] + (y_i - E[y_i|\mathbf{x}_i]) \\ &= F(\mathbf{x}_i'\boldsymbol{\beta}) + \varepsilon_i, \end{aligned}$$

where $\varepsilon_i = y_i - E[y_i|\mathbf{x}_i]$. Now assume that $F(\mathbf{x}_i^{'}\boldsymbol{\beta}) = \mathbf{x}_i'\boldsymbol{\beta}$, which produces the so-called *linear probability model.*

1. Show that $E[\varepsilon_i] = 0$ and $var(\varepsilon_i) = \mathbf{x}_i'\boldsymbol{\beta}(1 - \mathbf{x}_i'\boldsymbol{\beta})$.

2. Derive the GLS estimator of $\boldsymbol{\beta}$.

3. Explain how you would construct a feasible GLS estimator of $\boldsymbol{\beta}$ by consistently estimating $var(\varepsilon_i)$. Is it possible that your estimate of $var(\varepsilon_i)$ is negative? If so, what would you do to correct this problem?